

On a new class of rational cuspidal plane curves with two cusps

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Abstract

In this paper, we consider rational cuspidal plane curves having exactly two cusps whose complements have logarithmic Kodaira dimension two. We classify such curves with the property that the strict transforms of them via the minimal embedded resolution of the cusps have maximal self-intersection number.

1 Introduction

Let C be an algebraic curve on $\mathbf{P}^2 = \mathbf{P}^2(\mathbf{C})$. A singular point of C is said to be a *cuspidal* if it is a locally irreducible singular point. We say that C is *cuspidal* (resp. *bicuspidal*) if C has only cusps (resp. two cusps) as its singular points. For a cusp P of C , we denote the *multiplicity sequence* of (C, P) by $\overline{m}_P(C)$, or simply by \overline{m}_P . We usually omit the last 1's in \overline{m}_P . We use the abbreviation m_k for a subsequence of \overline{m}_P consisting of k consecutive m 's. For example, (2_k) means an A_{2k} singularity. The set of the multiplicity sequences of the cusps of a cuspidal plane curve C will be called the *numerical data* of C . For example, the rational quartic with three cusps has the numerical data $\{(2), (2), (2)\}$. We denote by $\bar{\kappa} = \bar{\kappa}(\mathbf{P}^2 \setminus C)$ the logarithmic Kodaira dimension of the complement $\mathbf{P}^2 \setminus C$.

Suppose that C is rational and bicuspidal. By [W, Ts], we have $\bar{\kappa} \geq 1$. Let C' denote the strict transform of C via the minimal embedded resolution of the cusps of C . We characterize rational bicuspidal plane curves C with $\bar{\kappa} = 1$ by $(C')^2$ in the following way.

Theorem 1. *If C is a rational bicuspidal plane curve, then $(C')^2 \leq 0$. Moreover, $(C')^2 = 0$ if and only if $\bar{\kappa} = 1$.*

We next consider rational bicuspidal plane curves C with $(C')^2 = -1$.

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Theorem 2. *The numerical data of a rational bicuspidal plane curve C with $(C')^2 = -1$ coincides with one of those in the following table, where a is a positive integer.*

No.	Numerical data	Degree
1	$\{(ab + b - 1, ab - 1, b_{a-1}, b - 1), ((ab)_2, b_a)\} \quad (b \geq 2)$	$2ab + b - 1$
2	$\{(ab + b, ab, b_a), ((ab + 1)_2, b_a)\} \quad (b \geq 2)$	$2ab + b + 1$
3	$\{(ab + 1, ab - b + 1, b_{a-1}), ((ab)_2, b_a)\} \quad (b \geq 3)$	$2ab + 1$
4	$\{(ab + b, ab, b_a), ((ab + b - 1)_2, b_a, b - 1)\} \quad (b \geq 3)$	$2ab + 2b - 1$

Conversely, for a given numerical data in the above table, there exists a rational cuspidal plane curve having that data.

In [Fe], many sequences of rational bicuspidal plane curves were constructed. The numerical data of the curves with $(C')^2 = -1$ among them coincide with the data 1, 2 and 3 with $a = 1$ in Theorem 2.

2 Preliminaries

Let D be a divisor on a smooth surface V , $\varphi : V' \rightarrow V$ a composite of successive blow-ups and $B \subset V'$ a divisor. We say that φ *contracts* B to D , or simply that B *shrinks to* D if $\varphi(\text{supp } B) = \text{supp } D$ and each center of blow-ups of φ is on D or one of its preimages. Let D_1, \dots, D_r be the irreducible components of D . We call D an *SNC-divisor* if D is a reduced effective divisor, each D_i is smooth, $D_i D_j \leq 1$ for distinct D_i, D_j , and $D_i \cap D_j \cap D_k = \emptyset$ for distinct D_i, D_j, D_k .

Assume that D is an SNC-divisor and that each D_i is projective. Let $\Gamma = \Gamma(D)$ denote the dual graph of D . We give the vertex corresponding to a component D_i the weight D_i^2 . We sometimes do not distinguish between D and its weighted dual graph Γ . We use the following notation and terminology (cf. [Fu, Section 3] and [MT1, Chapter 1]). A blow-up at a point $P \in D$ is said to be *sprouting* (resp. *subdivisional*) *with respect to* D if P is a smooth point (resp. node) of D . We also use this terminology for the case in which D is a point. By definition, the blow-up is subdivisional in this case. A component D_i is called a *branching component* of D if $D_i(D - D_i) \geq 3$.

Assume that Γ is connected and linear. In cases where $r > 1$, the weighted linear graph Γ together with a direction from an endpoint to the other is called a *linear chain*. By definition, the empty graph \emptyset and a weighted graph consisting of a single vertex without edges are linear chains. If necessary, renumber D_1, \dots, D_r so that the direction of the linear chain Γ is from D_1 to D_r and $D_i D_{i+1} = 1$ for $i = 1, \dots, r - 1$. We denote Γ by $[-D_1^2, \dots, -D_r^2]$. We sometimes write Γ as $[D_1, \dots, D_r]$. The linear chain

is called *rational* if every D_i is rational. In this paper, we always assume that every linear chain is rational. The linear chain Γ is called *admissible* if it is not empty and $D_i^2 \leq -2$ for each i . Set $r(\Gamma) = r$. We define the *discriminant* $d(\Gamma)$ of Γ as the determinant of the $r \times r$ matrix $(-D_i D_j)$. We set $d(\emptyset) = 1$.

Let $A = [a_1, \dots, a_r]$ be a linear chain. We use the following notation if $A \neq \emptyset$:

$${}^t A := [a_r, \dots, a_1], \quad \overline{A} := [a_2, \dots, a_r], \quad \underline{A} := [a_1, \dots, a_{r-1}].$$

The discriminant $d(A)$ has the following properties ([Fu, Lemma 3.6]).

Lemma 3. *Let $A = [a_1, \dots, a_r]$ be a linear chain.*

- (i) *If $r > 1$, then $d(A) = a_1 d(\overline{A}) - d(\overline{A}) = d({}^t A) = a_r d(\underline{A}) - d(\underline{A})$.*
- (ii) *If $r > 1$, then $d(\overline{A}) d(\underline{A}) - d(A) d(\overline{A}) = 1$.*
- (iii) *If A is admissible, then $\gcd(d(A), d(\overline{A})) = 1$ and $d(A) > d(\overline{A}) > 0$.*

Let $A = [a_1, \dots, a_r]$ be an admissible linear chain. The rational number $e(A) := d(\overline{A})/d(A)$ is called the *inductance* of A . By [Fu, Corollary 3.8], the function e defines a one-to-one correspondence between the set of all the admissible linear chains and the set of rational numbers in the interval $(0, 1)$. For a given admissible linear chain A , the admissible linear chain $A^* := e^{-1}(1 - e({}^t A))$ is called the *adjoint* of A ([Fu, 3.9]). Admissible linear chains and their adjoints have the following properties ([Fu, Corollary 3.7, Proposition 4.7]).

Lemma 4. *Let A and B be admissible linear chains.*

- (i) *If $e(A) + e(B) = 1$, then $d(A) = d(B)$ and $e({}^t A) + e({}^t B) = 1$.*
- (ii) *We have $A^{**} = A$, ${}^t(A^*) = ({}^t A)^*$ and $d(A) = d(A^*) = d(\overline{A^*}) + d(\underline{A})$.*
- (iii) *The linear chain $[A, 1, B]$ shrinks to $[0]$ if and only if $A = B^*$.*

For integers m, n with $n \geq 0$, we define $[m_n] = [\overbrace{m, \dots, m}^n]$, $t_n = [2_n]$. For non-empty linear chains $A = [a_1, \dots, a_r]$, $B = [b_1, \dots, b_s]$, we write $A * B = [\underline{A}, a_r + b_1 - 1, \overline{B}]$, $A^{*n} = \overbrace{A * \dots * A}^n$, where $n \geq 1$. We remark that $(A * B) * C = A * (B * C)$ for non-empty linear chains A, B and C . By using Lemma 3 and Lemma 4, we can show the following lemma.

Lemma 5. *Let $A = [a_1, \dots, a_r]$ be an admissible linear chain.*

- (i) *For a positive integer n , we have $[A, n + 1]^* = t_n * A^*$.*

(ii) We have $A^* = t_{a_r-1} * \cdots * t_{a_1-1}$.

(iii) If there exist positive integers m, n such that $[A, m+1] = [n+1, A]$ (resp. $A * t_m = t_n * A$), then $m = n$, $a_1 = \cdots = a_r = n+1$ (resp. $A = t_n^{*r(A^*)}$).

We will use the following lemma ([To, Corollary 8]).

Lemma 6. *Let a be a positive integer and A an admissible linear chain. Let B be a linear chain which is empty or admissible. Assume that a composite π of blow-downs contracts $[A, 1, B]$ to $[a]$ and that $[a]$ is the image of A under π .*

- (i) *The linear chain $[a]$ is the image of the first curve of A . There exists a positive integer n such that $A^* = [B, n+1, t_{a-1}]$. Moreover, $A = [a] * t_n * B^*$ if $B \neq \emptyset$.*
- (ii) *The first n blow-ups of π are sprouting and the remaining ones are subdivisional with respect to $[a]$ or its preimages. The composite of the subdivisional blow-ups contracts $[A, 1, B]$ to $[a] * t_n, 1$.*
- (iii) *The exceptional curve of each blow-up of π is a unique (-1) -curve in the preimage of $[a]$.*

*Conversely, $[a] * t_n * B^*, 1, B$ shrinks to $[a]$ for given positive integers a, n and an admissible linear chain B .*

2.1 Resolution of a cusp

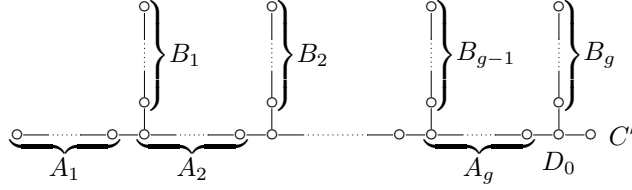
Let (C, P) be a curve germ on a smooth surface V . Suppose that (C, P) is a cusp. Let $\sigma : V' \rightarrow V$ be the minimal embedded resolution of (C, P) . That is, σ is the composite of the shortest sequence of blow-ups such that the strict transform C' of C intersects $\sigma^{-1}(P)$ transversally. Let $V' = V_n \xrightarrow{\sigma_{n-1}^{-1}} V_{n-1} \rightarrow \cdots \rightarrow V_2 \xrightarrow{\sigma_1} V_1 \xrightarrow{\sigma_0} V_0 = V$ be the blow-ups of σ . The following lemma follows from the assumptions that (C, P) is a cusp and σ is minimal.

Lemma 7. *For $i \geq 1$, the strict transform of C on V_i intersects $(\sigma_0 \circ \cdots \circ \sigma_{i-1})^{-1}(P)$ in one point, which is on the exceptional curve of σ_{i-1} . The point of intersection is the center of σ_i if $i < n$.*

Let D_0 denote the exceptional curve of the last blow-up of σ .

Lemma 8 ([To, Lemma 11]). *The following assertions hold.*

- (i) *The dual graph of $\sigma^{-1}(C)$ has the following shape, where $g \geq 1$ and A_1 contains the exceptional curve of σ_0 by definition.*



We number the irreducible components $A_{i,1}, A_{i,2}, \dots$ of A_i (resp. $B_{i,1}, B_{i,2}, \dots$ of B_i) from the left-hand side to the right (resp. the bottom to the top) in the above figure. With these directions and the weights $A_{i,1}^2, A_{i,2}^2, \dots, B_{i,1}^2, B_{i,2}^2, \dots$, we regard A_i, B_i as linear chains.

- (ii) The morphism σ can be written as $\sigma = \sigma_0 \circ \rho'_1 \circ \rho''_1 \circ \dots \circ \rho'_g \circ \rho''_g$, where each ρ'_i (resp. ρ''_i) consists of sprouting (resp. subdivisional) blow-ups of σ with respect to preimages of P .
- (iii) The morphisms $\rho_i := \rho'_i \circ \rho''_i$ have the following properties.
 - (a) For $j < i$, ρ_i does not change the linear chains A_j, B_j .
 - (b) For each i , $\rho_i \circ \dots \circ \rho_g$ maps $A_{i,1}$ to a (-1) -curve.
 - (c) ρ_g contracts the linear chain $A_g + D_0 + B_g$ to the (-1) -curve $\rho_g(A_{g,1})$. For $i < g$, ρ_i contracts the linear chain $(\rho_{i+1} \circ \dots \circ \rho_g)(A_i + A_{i+1,1} + B_i)$ to the (-1) -curve $(\rho_i \circ \dots \circ \rho_g)(A_{i,1})$.

We regard A_i and B_i as linear chains in the same way as in Lemma 8 (i). By Lemma 7, these linear chains are admissible. Let o_i denote the number of the blow-ups in ρ'_i . The following proposition follows from Lemma 6.

Proposition 9. *The following assertions hold for $i = 1, \dots, g$.*

- (i) We have $A_i = t_{o_i} * B_i^*$, $A_i^* = [B_i, o_i + 1]$.
- (ii) The linear chain A_i contains an irreducible component E with $E^2 \leq -3$.

2.2 The characteristic sequence of a cusp

Let the notation be as in the previous subsection. Put $\alpha_0 = \text{mult}_P C$. We take local coordinates (x, y) of V around $P = (0, 0)$ such that the germ (C, P) has a local parameterization:

$$x = t^{\alpha_0}, \quad y = \sum_{i=\alpha_1}^{\infty} c_i t^i \quad (c_{\alpha_1} \neq 0, \alpha_1 > \alpha_0, \alpha_1 \not\equiv 0 \pmod{\alpha_0}).$$

The *characteristic sequence* of (C, P) , which is denoted by $\text{Ch}_P = \text{Ch}_P(C)$, is a sequence $(\alpha_0, \alpha_1, \dots, \alpha_k)$ of positive integers defined by the following conditions.

(i) $\gcd(\alpha_0, \dots, \alpha_k) = 1$.

(ii) If $\gcd(\alpha_0, \dots, \alpha_{i-1}) > 1$, then α_i is the smallest j such that $c_j \neq 0$ and that $\gcd(\alpha_0, \dots, \alpha_{i-1}) > \gcd(\alpha_0, \dots, \alpha_{i-1}, j)$.

The multiplicity sequence of P is determined by Ch_P as follows. Put $\gamma_i = \alpha_i - \alpha_{i-1}$ for $i = 1, \dots, k$. Perform the Euclidean algorithm for $i = 1, \dots, k$:

$$\begin{aligned} \gamma_i &= a_{i,1}m_{i,1} + m_{i,2} & (0 < m_{i,2} < m_{i,1}), \\ m_{i,1} &= a_{i,2}m_{i,2} + m_{i,3} & (0 < m_{i,3} < m_{i,2}), \\ &\dots & \dots \\ m_{i,n_i-2} &= a_{i,n_i-1}m_{i,n_i-1} + m_{i,n_i} & (0 < m_{i,n_i} < m_{i,n_i-1}), \\ m_{i,n_i-1} &= a_{i,n_i}m_{i,n_i}, \end{aligned}$$

where $m_{1,1} = \alpha_0$ and $m_{i+1,1} = m_{i,n_i}$. Note that $a_{i,n_i} > 1$, $n_i > 1$, and that $a_{i,j} > 0$ if $j > 1$ but $a_{i,1} \geq 0$ for each i . The multiplicity sequence of P is given by

$$(\alpha_0, \overbrace{m_{1,1}, \dots, m_{1,1}}^{a_{1,1}}, \dots, \overbrace{m_{i,j}, \dots, m_{i,j}}^{a_{i,j}}, \dots, \overbrace{1, \dots, 1}^{a_{k,n_k}}).$$

Conversely, Ch_P is determined from \overline{m}_P by the above relation. See [BK, p.516, Theorem 12] for details, where γ_1 is defined as $\gamma_1 = \alpha_1$. We remark that the *Puiseux pairs* $(q_1, p_1), \dots, (q_k, p_k)$ of (C, P) are computed from Ch_P by the relations:

$$\alpha_0 = q_1 \cdots q_k, \quad \frac{\alpha_i}{\alpha_0} = \frac{p_i}{q_1 \cdots q_i}, \quad \gcd(q_i, p_i) = 1 \text{ for } i = 1, \dots, k.$$

We next describe the relation between the multiplicity sequence determined by Ch_P and the linear chains A_i, B_i .

Proposition 10 (cf. [BK, p.524, Theorem 15]). *We have the following relations between the multiplicity sequence $(m_{1,1}, (m_{1,1})_{a_{1,1}}, \dots, (m_{k,n_k})_{a_{k,n_k}})$ and $A_1, B_1, \dots, A_g, B_g$. In particular $g = k$.*

(i) *If n_i is an odd number, then*

$$\begin{aligned} A_i &= t_{a_{i,1}+1} * [a_{i,2}] * \cdots * t_{a_{i,n_i-2}+1} * [a_{i,n_i-1}] * t_{a_{i,n_i}}, \\ B_i &= [a_{i,n_i}] * t_{a_{i,n_i-1}+1} * \cdots * [a_{i,5}] * t_{a_{i,4}+1} * [a_{i,3}] * t_{a_{i,2}}, \end{aligned}$$

*where we interpret A_i, B_i as $A_i = t_{a_{i,1}+1} * [a_{i,2}] * t_{a_{i,3}}$, $B_i = [a_{i,3}] * t_{a_{i,2}}$ when $n_i = 3$.*

(ii) *If n_i is an even number, then*

$$\begin{aligned} A_i &= t_{a_{i,1}+1} * [a_{i,2}] * \cdots * t_{a_{i,n_i-1}+1} * [a_{i,n_i}], \\ B_i &= t_{a_{i,n_i}} * [a_{i,n_i-1}] * t_{a_{i,n_i-2}+1} * \cdots * [a_{i,5}] * t_{a_{i,4}+1} * [a_{i,3}] * t_{a_{i,2}}, \end{aligned}$$

*where we interpret A_i, B_i as $A_i = t_{a_{i,1}+1} * [a_{i,2}]$, $B_i = t_{a_{i,2}-1}$ when $n_i = 2$.*

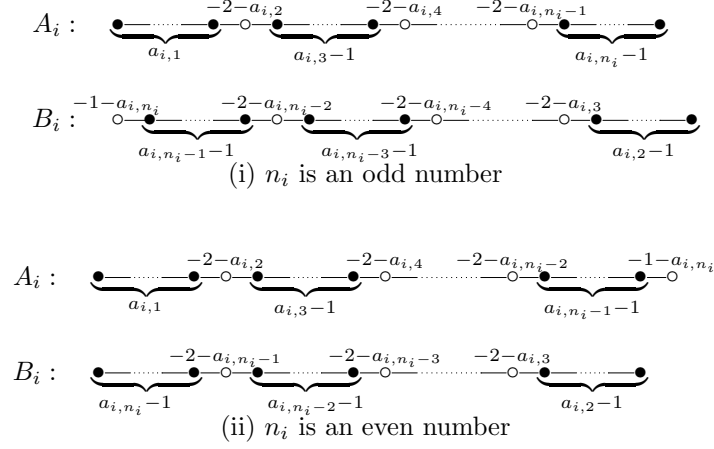


Figure 1: The weighted dual graphs of A_i and B_i

We have the weighted dual graphs in Figure 1 of A_i and B_i , where the vertices are ordered from the left-hand side to the right, and $*$ (resp. \bullet) denotes a (-1) -curve (resp. (-2) -curve).

In order to prove Proposition 10, we need Lemma 11 and Lemma 12 below. Let $V' = V_n \xrightarrow{\sigma_{n-1}} V_{n-1} \longrightarrow \cdots \longrightarrow V_2 \xrightarrow{\sigma_1} V_1 \xrightarrow{\sigma_0} V_0 = V$ be the blow-ups of the minimal embedded resolution σ of the cusp P as in the previous subsection. For $i > j$, put $\tau_{i,j} = \sigma_j \circ \sigma_{j+1} \circ \cdots \circ \sigma_{i-1} : V_i \rightarrow V_j$. Let E_i denote the exceptional curve of σ_{i-1} . We use the same symbol to denote the strict transforms of E_i . Let (C_i, P_i) denote the strict transform of the curve germ (C, P) on V_i , where $C_i \cap E_i = \{P_i\}$. Write $\overline{m}_P(C)$ as $\overline{m}_P(C) = (m_0, m_1, \dots)$.

Lemma 11 (cf. [FZ2, Lemma 1.3]). *Suppose $m_0 = \cdots = m_{q-1}$.*

(i) $(C_q E_q)_{P_q} = m_0$ and $(C_q E_i)_{P_q} = 0$ for each $i \neq q$.

(ii) *The dual graph of $\tau_{q,0}^{-1}(P)$ is linear. We have*

$$\tau_{q,0}^{-1}(P) = [E_1, E_2, \dots, E_q] = [t_{q-1}, 1].$$

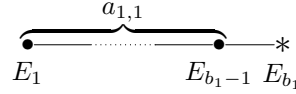
PROOF. The assertion (i) follows from [FZ2, Lemma 1.3]. We prove the assertion (ii) by induction on q . The assertion is clear if $q = 1$. Assume $q > 1$. We have $\overline{m}_{P_1} = (m_1, m_2, \dots)$. By the induction hypothesis, the dual graph of $\tau_{q,1}^{-1}(P_1)$ is linear and $\tau_{q,1}^{-1}(P_1) = [E_2, \dots, E_q] = [t_{q-2}, 1]$. By (i), the center of σ_1 is on E_1 , while that of σ_i is not for $i \geq 2$. This means that $E_1^2 = -2$ on V_q and that E_1 intersects only E_2 among E_2, \dots, E_q . \square

Lemma 12 (cf. [FZ2, Lemma 1.4]). *Let l be a projective curve on V which is smooth at P . Let l_i denote the strict transform of l on V_i . Write $(Cl)_P = qm_0 + r$, where $1 \leq q$, $0 \leq r < m_0$.*

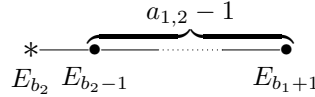
- (i) *We have $m_0 = \dots = m_{q-1}$. Moreover, $m_q = r$ if $r > 0$.*
- (ii) *We have $(C_q l_q)_{P_q} = r$, $l_q^2 = l^2 - q$, $\tau_{q,0}^{-1}(l) = E_1 + \dots + E_q + l_q$, $E_q l_q = 1$ and $E_1 l_q = \dots = E_{q-1} l_q = 0$.*

PROOF. The assertion (i) follows from [FZ2, Lemma 1.4]. We prove the assertion (ii) by induction on q . On V_1 , we have $(C_1 l_1)_{P_1} = (q-1)m_0 + r$, $l_1^2 = l^2 - 1$ and $\sigma_0^{-1}(l) = E_1 + l_1$. So the assertion is clear if $q = 1$. Assume that $q > 1$. We use the induction hypothesis on V_1 . Since $(C_1 l_1)_{P_1} = (q-1)m_1 + r$, we have $(C_q l_q)_{P_q} = r$, $l_q^2 = l_1^2 - q + 1 = l^2 - q$, $\tau_{q,1}^{-1}(l_1) = E_2 + \dots + E_q + l_q$, $E_q l_q = 1$ and $E_2 l_q = \dots = E_{q-1} l_q = 0$. Since $E_1 l_1 = 1$ on V_1 , the curve E_1 does not intersect l_q . \square

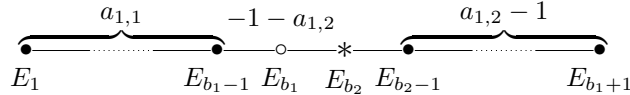
Proof of Proposition 10. We first show the assertion for A_1 and B_1 by induction on n_1 . Put $b_i = 1 + \sum_{j=1}^i a_{1,j}$. By applying Lemma 11 to (C, P) with $q = b_1$, we have $\tau_{b_1,0}^{-1}(P) = [E_1, \dots, E_{b_1-1}, E_{b_1}] = [t_{a_{1,1}}, 1]$ and $(C_{b_1} E_{b_1})_{P_{b_1}} = m_{1,1}$. We see $\overline{m}_{P_{b_1}}(C_{b_1}) = ((m_{1,2})_{a_{1,2}}, \dots)$.



We next apply Lemma 11 to (C_{b_1}, P_{b_1}) with $q = a_{1,2}$. We have $\tau_{b_2,b_1}^{-1}(P_{b_1}) = [E_{b_1+1}, \dots, E_{b_2-1}, E_{b_2}] = [t_{a_{1,2}}, 1]$ and $(C_{b_2} E_{b_2})_{P_{b_2}} = m_{1,2}$.



We then apply Lemma 12 to E_{b_1} and (C_{b_1}, P_{b_1}) . Because $(C_{b_1} E_{b_1})_{P_{b_1}} = a_{1,2}m_{1,2}$ ($n_1 = 2$) or $(C_{b_1} E_{b_1})_{P_{b_1}} = a_{1,2}m_{1,2} + m_{1,3}$ ($n_1 > 2$), it follows that $\tau_{b_2,b_1}^{-1}(E_{b_1}) = [E_{b_1+1}, \dots, E_{b_2-1}, E_{b_2}, E_{b_1}] = [t_{a_{1,2}-1}, 1, 1 + a_{1,2}]$ and that $(C_{b_2} E_{b_1})_{P_{b_2}} = 0$ ($n_1 = 2$) or $(C_{b_2} E_{b_1})_{P_{b_2}} = m_{1,3}$ ($n_1 > 2$). Since $P_{b_1} \notin E_i$ for $i < b_1$, we see $\tau_{b_2,0}^{-1}(P) = [E_1, \dots, E_{b_1-1}, E_{b_1}, E_{b_2}, E_{b_2-1}, \dots, E_{b_1+1}] = [t_{a_{1,1}}, 1 + a_{1,2}, 1, t_{a_{1,2}-1}]$.



Suppose that $n_1 = 2$. Since $m_{1,1} = a_{1,2}m_{1,2}$, we have $P_{b_2} \notin E_i$ for $i < b_2$. Thus the weighted dual graph of $\tau_{b_2,0}^{-1}(P) - E_{b_2}$ is unchanged by the remaining blow-ups. The vertex corresponding to E_{b_2} is a branching

component of the dual graph of $\sigma^{-1}(P) + C'$. Because A_1 contains E_1 , we have $A_1 = t_{a_{1,1}+1} * [a_{1,2}]$, $B_1 = t_{a_{1,2}-1}$.

Suppose that $n_1 > 2$. We have $\overline{m}_{P_{b_1}}(C_{b_1}) = ((m_{1,2})_{a_{1,2}}, m_{1,3}, \dots)$. Since $(C_{b_2}E_{b_1})_{P_{b_2}} = m_{1,3} = \text{mult}_{P_{b_2}}(C_{b_2})$, we see $P_{b_2} \in E_{b_1}$ and $P_i \notin E_{b_1}$ for $i > b_2$. It follows that $E_{b_1}^2 = -a_{1,2} - 2$ and that E_{b_1} intersects E_{b_2+1} on V_i for $i > b_2$. We apply the induction hypothesis to (C_{b_1}, P_{b_1}) . Put $T = \tau_{b_{n_1}, b_1}^{-1}(P_{b_1})$. We write it as $T = [A, 1, B]$, where A contains E_{b_1+1} . If n_1 is an odd number, then

$$\begin{aligned} A &= t_{a_{1,2}} * [a_{1,3}] * t_{a_{1,4}+1} * [a_{1,5}] * \cdots * t_{a_{1,n_1-1}+1} * [a_{1,n_1}], \\ B &= t_{a_{1,n_1}} * [a_{1,n_1-1}] * t_{a_{1,n_1-2}+1} * \cdots * [a_{1,6}] * t_{a_{1,5}+1} * [a_{1,4}] * t_{a_{1,3}}. \end{aligned}$$

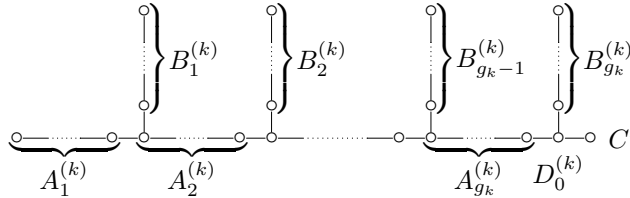
If n_1 is an even number, then

$$\begin{aligned} A &= t_{a_{1,2}} * [a_{1,3}] * t_{a_{1,4}+1} * [a_{1,5}] * \cdots * t_{a_{1,n_1-2}+1} * [a_{1,n_1-1}] * t_{a_{1,n_1}}, \\ B &= [a_{1,n_1}] * t_{a_{1,n_1-1}+1} * \cdots * [a_{1,6}] * t_{a_{1,5}+1} * [a_{1,4}] * t_{a_{1,3}}. \end{aligned}$$

The first curve of A is E_{b_1+1} by Lemma 8 (iii). It follows that $\tau_{b_{n_1}, 0}^{-1}(P) = [E_1, \dots, E_{b_1-1}, E_{b_1}, {}^tT]$. By the induction hypothesis, A and B are unchanged by the remaining blow-ups. We infer that $\tau_{b_{n_1}, 0}^{-1}(P) - E_{b_{n_1}}$ is also unchanged by the remaining blow-ups. Hence $A_1 = [t_{a_{1,1}}, a_{1,2} + 2, {}^tB]$, $B_1 = {}^tA$. We can prove the assertion for A_i and B_i with $i \geq 2$ by using the same arguments as above, where (C_b, P_b) ($b = \sum_{j=1}^{i-1} \sum_{k=1}^{n_j} a_{j,k}$) plays the role of (C, P) . \square

3 Proof of Theorem 1

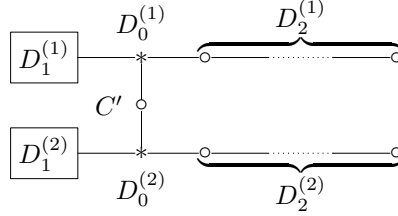
Let C be a rational bicuspidal plane curve. Let P_1, P_2 denote the cusps of C . Let $\sigma : V \rightarrow \mathbf{P}^2$ be the minimal embedded resolution of the cusps and C' the strict transform of C via σ . Put $D := \sigma^{-1}(C)$. We may assume $\sigma = \sigma^{(1)} \circ \sigma^{(2)}$, where $\sigma^{(k)}$ consists of the blow-ups over P_k . We decompose the dual graph of $\sigma^{-1}(P_k)$ ($k = 1, 2$) into subgraphs $A_1^{(k)}, B_1^{(k)}, \dots, A_{g_k}^{(k)}, B_{g_k}^{(k)}, D_0^{(k)}$ in the same way as in Lemma 8.



By definition, $A_1^{(k)}$ contains the exceptional curve of the first blow-up over P_k . We give the weighted graphs $A_1^{(k)}, \dots, A_{g_k}^{(k)}$ (resp. $B_1^{(k)}, \dots, B_{g_k}^{(k)}$) the direction from the left-hand side to the right (resp. from the bottom to the

top) of the above figure. With these directions, we regard $A_i^{(k)}$ and $B_i^{(k)}$ as linear chains. Let $\sigma_0^{(k)}$ denote the first blow-up of $\sigma^{(k)}$. By Lemma 8, there exists a decomposition $\sigma^{(k)} = \sigma_0^{(k)} \circ \sigma_{1,1}^{(k)} \circ \sigma_{1,2}^{(k)} \circ \cdots \circ \sigma_{g_k,1}^{(k)} \circ \sigma_{g_k,2}^{(k)}$ such that each $\sigma_{i,1}^{(k)}$ (resp. $\sigma_{i,2}^{(k)}$) consists of sprouting (resp. subdivisional) blow-ups with respect to preimages of P_k . The morphism $\sigma_{i,1}^{(k)} \circ \sigma_{i,2}^{(k)}$ contracts $[A_i^{(k)}, 1, B_i^{(k)}]$ to a (-1) -curve for $i \geq 1$. Let $o_i^{(k)}$ denote the number of the blow-ups of $\sigma_{i,1}^{(k)}$.

We first show the “if” part of Theorem 1. Assume that $\bar{\kappa}(\mathbf{P}^2 \setminus C) = 1$. Put $D_1^{(k)} = B_{g_k}^{(k)}$ and $D_2^{(k)} = D^{(k)} - (D_0^{(k)} + D_1^{(k)})$. The dual graph of D has the following shape.



Following [FZ1], we consider a strictly minimal model (\tilde{V}, \tilde{D}) of (V, D) . We successively contract (-1) -curves E satisfying one of the following conditions: (1) $E \subset D$ and $(D - E)E = 0$, (2) $E \subset D$ and $(D - E)E = 1$, (3) $E \subset D$ and $(D - E)E = 2$, (4) $E \not\subset D$ and $DE = 0$, (5) $E \not\subset D$ and $DE = 1$. After a finite number of contractions, we have no (-1) -curves satisfying the above conditions. Let $\pi : V \rightarrow \tilde{V}$ be the composite of the contractions.

Lemma 13. *The morphism π does not contract irreducible curves meeting with C' . In particular, $(C')^2 = -1$ if and only if C' is contracted by π .*

PROOF. Suppose that there exists an irreducible curve E on V which intersects C' and is contracted by π . If E is a component of D , then E is either $D_0^{(1)}$ or $D_0^{(2)}$. Since E is a (-1) -curve, we may assume that π contracts E first. But this contraction is not allowed, since $(D - E)E = 3$. Thus $E \not\subset D$. Since E is contracted by π , E does not intersect any components of D other than C' . This means that $\sigma(E)$ is a plane curve with $\sigma(E)^2 \leq -1$, which is impossible. \square

For a divisor E on V , we write $\tilde{E} = \pi_*(E)$. It is clear that \tilde{D} is an SNC-divisor and $\bar{\kappa}(\tilde{V} \setminus \tilde{D}) = 1$.

Lemma 14. *There exists a fibration $\tilde{p} : \tilde{V} \rightarrow \mathbf{P}^1$ whose general fiber F is \mathbf{P}^1 and $\tilde{D}F = 2$.*

PROOF. By [Ka, Theorem 2.3] and the fact that $\tilde{V} \setminus \tilde{D}$ is affine, there exists a fibration $\tilde{p} : \tilde{V} \rightarrow W$ over a smooth curve W whose general fiber F is \mathbf{P}^1 and $\tilde{D}F = 2$. Since $q(\tilde{V}) = 0$, the curve W must be \mathbf{P}^1 . \square

The fibration \tilde{p} is obtained from a \mathbf{P}^1 -bundle $\hat{p} : \Sigma \rightarrow \mathbf{P}^1$ by successive blow-ups $\tilde{\pi} : \tilde{V} \rightarrow \Sigma$. Putting $p = \tilde{p} \circ \pi$, we have the following commutative diagram.

$$\begin{array}{ccccc} V & \xrightarrow{\pi} & \tilde{V} & \xrightarrow{\tilde{\pi}} & \Sigma \\ & \searrow p & \searrow \tilde{p} & & \downarrow \hat{p} \\ & & & & \mathbf{P}^1 \end{array}$$

Following [FZ1], we use the following terminology. The triple $(\tilde{V}, \tilde{D}, \tilde{p})$ is called a \mathbf{C}^* -triple. A component of \tilde{D} is called *horizontal* if the image of it under \tilde{p} is 1-dimensional. Let \tilde{H} be the sum of the horizontal components of $(\tilde{V}, \tilde{D}, \tilde{p})$. The \mathbf{C}^* -triple $(\tilde{V}, \tilde{D}, \tilde{p})$ is called of *twisted type* if \tilde{H} is irreducible; otherwise it is called of *untwisted type*. A fiber of \tilde{p} is called a *full fiber* of $(\tilde{V}, \tilde{D}, \tilde{p})$ if it is contained in \tilde{D} . Let f denote the number of the full fibers of $(\tilde{V}, \tilde{D}, \tilde{p})$.

Lemma 15. *The \mathbf{C}^* -triple has the following properties.*

- (i) *The \mathbf{C}^* -triple is of untwisted type.*
- (ii) *We have $f \leq 1$. The fibration \tilde{p} has at least two singular fibers.*
- (iii) *The weighted dual graph of a singular fiber of \tilde{p} is a linear chain $[A, 1, B]$, where A, B are admissible and are connected components of $\tilde{D} - \tilde{H}$. The curve \tilde{H} intersects only the first vertex of A and the last of B .*

PROOF. By [Ki, Theorem 3], the \mathbf{C}^* -triple is of untwisted type. The assertions (ii), (iii) follow from [FZ1, Lemma 4.4, Theorem 5.8 and 5.11]. \square

Lemma 13 and the assertion (i) of the following proposition show the “if” part of Theorem 1.

Proposition 16. *The following assertions hold.*

- (i) *We have $\tilde{H} = \tilde{D}_0^{(1)} + \tilde{D}_0^{(2)}$. The curve \tilde{C}' is a full fiber of \tilde{p} .*
- (ii) *The fibration \tilde{p} has exactly two singular fibers $\tilde{F}_1 = \tilde{D}_1^{(1)} + \tilde{E}_1 + \tilde{D}_a^{(2)}$, $\tilde{F}_2 = \tilde{D}_2^{(1)} + \tilde{E}_2 + \tilde{D}_b^{(2)}$, where $\{a, b\} = \{1, 2\}$ and \tilde{E}_i is the (-1) -curve in \tilde{F}_i .*

PROOF. We first show that π does not contract C' . Assume the contrary. Since $(\tilde{D}_0^{(1)})^2 \geq 0$, $\tilde{D}_0^{(1)}$ is either a horizontal component or a full fiber. Assume $\tilde{D}_0^{(1)}$ is a full fiber. Since $(\tilde{D} - \tilde{D}_0^{(1)})\tilde{D}_0^{(1)} = 2$, one of $\tilde{D}_1^{(1)}$ or $\tilde{D}_2^{(1)}$ is contracted by π to a point on $\tilde{D}_0^{(1)}$. Thus we have $(\tilde{D}_0^{(1)})^2 > 0$, which is

a contradiction. Similarly, $\tilde{D}_0^{(2)}$ is not a full fiber. Thus $\tilde{H} = \tilde{D}_0^{(1)} + \tilde{D}_0^{(2)}$. Let \tilde{F} be the fiber of \tilde{p} passing through the point of intersection of $\tilde{D}_0^{(1)}$ and $\tilde{D}_0^{(2)}$. The strict transform F of \tilde{F} in V intersects only C' among the irreducible components of D . Hence $\sigma(F)$ is a plane curve with $\sigma(F)^2 < 0$, which is a contradiction. Thus π does not contract C' .

Since $(\tilde{D}_0^{(1)})^2 \geq -1$, $\tilde{D}_0^{(1)}$ is either a horizontal component or a full fiber. Suppose that $\tilde{D}_0^{(1)}$ is a full fiber. Then \tilde{C}' must be a horizontal component. This means that \tilde{p} has at most one singular fiber, which contradicts Lemma 15. Thus $\tilde{D}_0^{(1)}$ is a horizontal component. Similarly, $\tilde{D}_0^{(2)}$ is a horizontal component. Hence \tilde{C}' must be a full fiber of \tilde{p} . The assertion (ii) follows from (i) and Lemma 15. \square

We prove the remaining assertions of Theorem 1. Let C be a rational bicuspidal plane curve. Suppose $(C')^2 \geq 0$. Since $\dim |C'| = 1 + (C')^2$, it follows that $\mathbf{P}^2 \setminus C$ contains a surface $\mathbf{C}^* \times B$, where B is a curve. Hence we have $\bar{\kappa}(\mathbf{P}^2 \setminus C) \leq 1$. By [W], $\bar{\kappa}(\mathbf{P}^2 \setminus C) \geq 0$. By [Ts, Proposition 1], $\bar{\kappa}(\mathbf{P}^2 \setminus C) \geq 1$. See also [Ko, O]. Hence we have $\bar{\kappa}(\mathbf{P}^2 \setminus C) = 1$ and $(C')^2 = 0$.

4 Proof of Theorem 2

Let C be a rational bicuspidal plane curve. Let P_1, P_2 denote the cusps of C . Let $\sigma : V \rightarrow \mathbf{P}^2$ be the minimal embedded resolution of the cusps. Let C', D , etc. have the same meaning as in the first paragraph of the previous section. Assume that $(C')^2 = -1$. Put $F'_0 = D_0^{(1)}$. Let $\sigma' : V \rightarrow V'$ be the contraction of C' . Since $(F'_0)^2 = 0$ on V' , there exists a \mathbf{P}^1 -fibration $p' : V' \rightarrow \mathbf{P}^1$ such that F'_0 is a nonsingular fiber. Put $p = p' \circ \sigma' : V \rightarrow \mathbf{P}^1$ and $F_0 = F'_0 + C'$.

Remark. Since $(D_0^{(2)})^2 = 0$ on V' , there exists another \mathbf{P}^1 -fibration such that $D_0^{(2)}$ is a nonsingular fiber.

The surface $X = V \setminus D$ is a \mathbf{Q} -homology plane. Namely $h^i(X, \mathbf{Q}) = 0$ for $i > 0$. A general fiber of $p|_X$ is a curve $\mathbf{C}^{**} = \mathbf{P}^1 \setminus \{3 \text{ points}\}$. Such fibrations have already been classified in [MiSu]. We will use their result to prove our theorem. There exists a birational morphism $\varphi : V \rightarrow \Sigma_n$ from V onto the Hirzebruch surface Σ_n of degree n for some n such that $p \circ \varphi^{-1} : \Sigma_n \rightarrow \mathbf{P}^1$ is a \mathbf{P}^1 -bundle. The morphism φ is the composite of the successive contractions of the (-1) -curves in the singular fibers of p . Let S_1 and S_3 be the irreducible components of $A_{g_1}^{(1)} + B_{g_1}^{(1)}$ meeting with $D_0^{(1)}$. Put $S_2 = D_0^{(2)}$. The curves S_1, S_2 and S_3 are 1-sections of p . The divisor D contains no other sections of p .

Lemma 17. *We may assume that $\varphi(S_1 + S_2 + S_3)$ is smooth. We have $\varphi(S_1) \sim \varphi(S_2) \sim \varphi(S_3)$ (linearly equivalent) and $n = \varphi(S_i)^2 = 0$ for each i .*

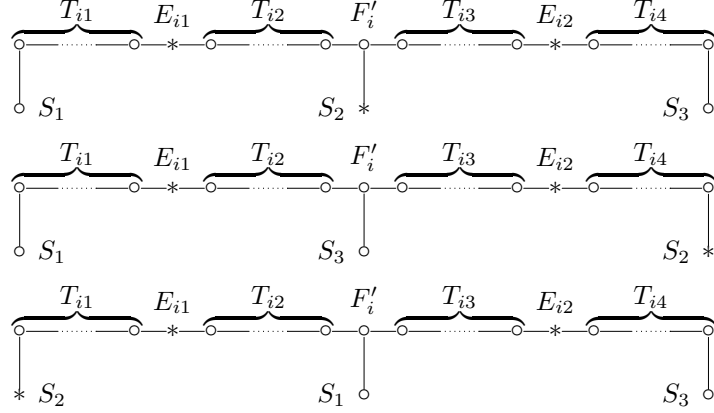


Figure 2: Candidates for the dual graph of F_i , $i = 1, 2$

PROOF. We only prove the first assertion. Suppose $\varphi(S_1 + S_2 + S_3)$ has a singular point P . Let ϕ_1 be the blow-up at P . Since $S_1 + S_2 + S_3$ is smooth on V , we can arrange the order of the blow-ups of φ so that $\varphi = \phi_1 \circ \varphi'$. Let F' be the strict transform via ϕ_1 of the fiber of $p \circ \varphi'^{-1}$ passing through P . Let ϕ_2 be the contraction of F' . Since F' is an irreducible component of a singular fiber of $p \circ \varphi'^{-1}$, we can replace φ with $\phi_2 \circ \varphi'$. We infer that P can be resolved by repeating the above process. Hence we may assume that $\varphi(S_1 + S_2 + S_3)$ is smooth. \square

Let F_1, \dots, F_l be all singular fibers of p other than F_0 . For $i = 1, \dots, l$, let E_i be the sum of the irreducible components of F_i which are not components of D . Since D contains no loop, each E_i is not empty. It follows that the base curve of the \mathbf{C}^{**} -fibration $p|_X$ is \mathbf{C} . Because $\bar{\kappa}(V \setminus D) = 2$, each irreducible component of E_i meets with D in at least two points by [MT2, Main Theorem]. In [MiSu, Lemma 1.5], singular fibers of a \mathbf{C}^{**} -fibration with three 1-sections were classified into several types. Among them, only singular fibers of type (I₁) and (III₁) satisfy the conditions that each irreducible component of E_i meets with D in at least two points. From the fact that D contains no loop, we infer that each F_i is of type (III₁). By [MiSu, Lemma 2.3], p has at most two singular fibers other than F_0 . Since S_2 meets with $D - S_2$ in three points, p has exactly three singular fibers F_0, F_1 and F_2 . For $i = 1, 2$, the dual graph of $F_i + S_1 + S_2 + S_3$ coincides with one of those in Figure 2, where $*$ denotes a (-1) -curve and $E_i = E_{i1} + E_{i2}$. The graph $T_{i,j}$ may be empty for each j .

Lemma 18. *We have $\varphi(F_i) = \varphi(F'_i)$ for $i = 0, 1, 2$. For $i = 1, 2$, the dual graph of $F_i + S_1 + S_2 + S_3$ must be the first one in Figure 2.*

PROOF. By Lemma 17, we have $\varphi(F_0) = \varphi(F'_0)$. Suppose that φ contracts F'_1 . Let F'_1 intersect S_j . Write φ as $\varphi = \varphi_3 \circ \varphi_2 \circ \varphi_1$, where φ_2 is the

contraction of F'_1 . The curve $\varphi_1(F'_1)$ intersects three irreducible components of $D + E_1 + E_2$. By Lemma 17, $\varphi_1(F'_1)$ does not intersect the images under φ_1 of sections other than S_j . It follows that $\varphi_2(\varphi_1(S_j))\varphi_2(\varphi_1(F_1)) > 1$, which is absurd. Thus φ does not contract F'_1 . Similarly, φ does not contract F'_2 . If one of F'_1, F'_2 does not intersect S_2 , then $\varphi(S_2)^2 > 0$, which contradicts Lemma 17. Thus F'_1 and F'_2 intersect S_2 . \square

By Lemma 18, the dual graph of $D + E_1 + E_2$ must coincide with that in Figure 3.

5 Proof of Theorem 2 — continued

Let the notation be as in the previous section. We infer $g_i \leq 2$ for $i = 1, 2$. With the direction from the left-hand side to the right of Figure 3, we regard T_{ij} 's as linear chains. Put $s_i = -S_i^2$ and $f_j = -(F'_j)^2$ for $i \neq 2$ and $j = 1, 2$. We have $s_i \geq 2$ and $f_j \geq 2$.

Lemma 19. *The following assertions hold.*

- (i) *We may assume $B_{g_1}^{(1)} = [S_1, T_{11}]$. We have $T_{21} = \emptyset$. There exists a non-negative integer l_{22} such that $T_{22} = t_{l_{22}}$.*
- (ii) *There exist positive integers k_{12}, k_{34} such that $[S_1, T_{11}]^* = [T_{12}, k_{12} + 1, t_{l_{22}}]$, $[F'_1, T_{13}]^* = [T_{14}, k_{34} + 1, t_{k_{12}-1}]$ and $[T_{24}, S_3]^* = [t_{k_{34}-1}, f_2, T_{23}]$. We have $A_{g_1}^{(1)} = t_{o_{g_1}^{(1)}} * [T_{12}, k_{12} + 1, t_{l_{22}}]$.*

PROOF. (i) We may assume $B_{g_1}^{(1)} = [S_1, T_{11}]$ because the dual graph of $D + E_1 + E_2$ is symmetric about the line passing through F'_1, S_2 and F'_2 in Figure 3, and the line passing through S_1, S_2 and S_3 . We have $T_{21} = \emptyset$. If $T_{22} \neq \emptyset$, then φ contracts $[E_{21}, T_{22}]$ to a (-1) -curve. By Lemma 6, there exists a positive integer l_{22} such that $T_{22} = t_{l_{22}}$. We set $l_{22} = 0$ if $T_{22} = \emptyset$.

(ii) We may assume that $\varphi = \varphi_0 \circ \varphi_{21} \circ \varphi_{11} \circ \varphi_{12} \circ \varphi_{22}$, where φ_0 contracts C' and φ_{ij} contracts $T_{i,2j-1} + E_{ij} + T_{i,2j}$ to a point. Since $[E_{21}, T_{22}] = [1, t_{l_{22}}]$ and $\varphi(S_1)^2 = 0$, φ_{11} contracts $[S_1, T_{11}, E_{11}, T_{12}]$ to $[l_{22} + 1]$ by Lemma 4 (iii). By Lemma 6, there exists a positive integer k_{12} such that $[S_1, T_{11}]^* = [T_{12}, k_{12} + 1, t_{l_{22}}]$. The composite of the subdivisional blow-ups of φ_{11} with respect to the preimages of S_1 contracts $[T_{11}, E_{11}, T_{12}]$ to $[t_{k_{12}-1}, 1]$. Since $\varphi(F'_1)^2 = 0$, φ_{12} contracts $[F'_1, T_{13}, E_{12}, T_{14}]$ to $[k_{12}]$ by Lemma 4 (iii). By Lemma 6, there exists a positive integer k_{34} such that $[F'_1, T_{13}]^* = [T_{14}, k_{34} + 1, t_{k_{12}-1}]$. Similarly, φ_{22} contracts $[T_{23}, E_{22}, T_{24}, S_3]$ to $[k_{34}]$. By Lemma 6, there exists a positive integer k such that $[S_3, {}^tT_{24}]^* = [{}^tT_{23}, k + 1, t_{k_{34}-1}]$. Since $\varphi(F'_2)^2 = 0$, we have $0 = -f_2 + k + 1$. The last assertion follows from Proposition 9. \square

Now we prove Theorem 2. The linear chain $B_{g_2}^{(2)}$ coincides with one of ${}^t[T_{12}, F'_1]$, $[F'_1, T_{13}]$, ${}^t[T_{22}, F'_2]$ or $[F'_2, T_{23}]$.

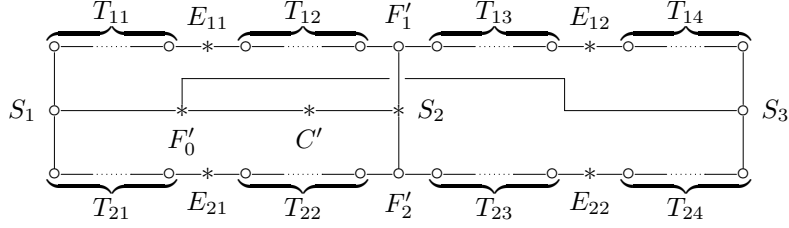


Figure 3: The dual graph of $D + E_1 + E_2$

5.1 $B_{g_2}^{(2)} = {}^t[T_{12}, F_1']$

Lemma 20. *We have $T_{13} = \emptyset$, $k_{34} = 1$, $T_{14} = t_{f_1-k_{12}-1}$ and $[T_{24}, S_3]^* = [f_2, T_{23}]$.*

PROOF. It is clear that $T_{13} = \emptyset$. By Lemma 19, we see $[T_{14}, k_{34}+1, t_{k_{12}-1}] = t_{f_1-1}$. This means that $k_{34} = 1$, $T_{14} = t_{f_1-k_{12}-1}$. By Lemma 19, we have $[T_{24}, S_3]^* = [f_2, T_{23}]$. \square

Case (1): $g_2 = 1$. Either $A_1^{(2)} = [T_{22}, F_2']$ or $A_1^{(2)} = {}^t[F_2', T_{23}]$. Suppose $A_1^{(2)} = [T_{22}, F_2']$. We have $T_{23} = \emptyset$. By Lemma 20, we get $[T_{24}, S_3] = t_{f_2-1}$. Thus $T_{14} + S_3 + T_{24}$ consists of (-2) -curves and so does $A_{g_1}^{(1)}$, which contradicts Proposition 9. Hence $A_1^{(2)} = {}^t[F_2', T_{23}]$. It follows that $T_{22} = \emptyset$. By Lemma 20 and Proposition 9, we obtain $[T_{24}, S_3] = [o_1^{(2)}+1, T_{12}, F_1']$. We have $T_{24} \neq \emptyset$.

Suppose $T_{14} = \emptyset$. We have $f_1 = k_{12}+1$, $g_1 = 1$ and $A_1^{(1)} = [T_{24}, S_3]$. Since $t_{l_{22}} = T_{22} = \emptyset$, we get $B_1^{(1)} = [T_{12}, f_1]^*$ by Lemma 19. By Proposition 9, $[o_1^{(2)}+1, T_{12}, f_1] = [T_{24}, S_3] = t_{o_1^{(1)}} * [T_{12}, f_1]$. Thus $o_1^{(1)} = 2$, $o_1^{(2)} = 1$. Since $[T_{12}, f_1] = t_1 * [T_{12}, f_1]$, we infer $[T_{12}, f_1]^* = [[T_{12}, f_1]^*, 2]$, which is absurd. Hence $T_{14} \neq \emptyset$. If $g_1 = 1$, then $A_1^{(1)} = [T_{14}, s_3]$ and $T_{24} = \emptyset$, which is a contradiction. Thus $g_1 = 2$. Since $A_2^{(1)} = S_3$, we have $T_{12} = \emptyset$, $o_2^{(1)} = 1$ and $A_2^{(1)} = S_3 = [k_{12}+2]$ by Lemma 19. By Proposition 9, $B_2^{(1)} = t_{k_{12}}$.

By Lemma 20, T_{14} consists of (-2) -curves. By Proposition 9, $B_1^{(1)} = {}^tT_{14}$ and $A_1^{(1)} = T_{24}$. By Lemma 20 and Proposition 9, we get $[f_2, T_{23}] = t_{k_{12}+1} * [t_{f_1-k_{12}-1}, o_1^{(1)}+1]$. Since $\emptyset \neq T_{14} = t_{f_1-k_{12}-1}$, we have $[f_2, T_{23}] = [t_{k_{12}}, 3, t_{f_1-k_{12}-2}, o_1^{(1)}+1]$. On the other hand, $[f_2, T_{23}] = {}^tA_1^{(2)} = t_{f_1-1} * t_{o_1^{(2)}} = [t_{f_1-2}, 3, t_{o_1^{(2)}}-1]$ by Proposition 9. This means that $o_1^{(2)} = 2$, $o_1^{(1)} = 1$ and $f_1 = k_{12}+2$. Write $k = k_{12}$. We have $A_2^{(1)} = [k+2]$, $B_2^{(1)} = t_k$, $B_1^{(1)} = [2]$ and $B_1^{(2)} = [k+2]$. By Proposition 9, we see $A_1^{(1)} = [3]$ and $A_1^{(2)} = [2, 3, t_k]$. It follows from Proposition 10 that the numerical data of

C is equal to $\{(2(k+1), (k+1)_2), ((k+2)_2, k+1)\}$, which coincides with the data 2 with $a = 1$, $b = k + 1$.

Case (2): $g_2 = 2$. By Lemma 19, T_{22} consists of (-2) -curves. By Proposition 9, we have $B_1^{(2)} = {}^tT_{22}$ and $A_1^{(2)} = {}^tT_{23}$. Thus $A_1^{(2)} = t_{o_1^{(2)}} * [l_{22} + 1]$. Since $T_{22} \neq \emptyset$, we infer $r(A_{g_1}^{(1)}) \geq 2$ by Lemma 19. It follows that $g_1 = 1$. Either $A_1^{(1)} = [T_{14}, S_3]$ or $A_1^{(1)} = [T_{24}, S_3]$. By Lemma 20, T_{14} consists of (-2) -curves. By Lemma 19, $A_1^{(1)} - S_3$ contains an irreducible component other than a (-2) -curve. Thus $A_1^{(1)} = [T_{24}, S_3]$, $T_{14} = \emptyset$ and $f_1 = k_{12} + 1$.

Because $A_2^{(2)} = F'_2$, we have $t_{f_2-1} = [o_2^{(2)} + 1, T_{12}, f_1]$ by Proposition 9. This shows that $o_2^{(2)} = 1$, $f_1 = 2$ and $T_{12} = t_{f_2-3}$. Thus $k_{12} = 1$. By Lemma 19, $[S_1, T_{11}] = t_{f_2+l_{22}-2}^* = [f_2 + l_{22} - 1]$. By Proposition 9 and Lemma 20, $[f_2, T_{23}] = A_1^{(1)*} = [S_1, T_{11}, o_1^{(1)} + 1] = [f_2 + l_{22} - 1, o_1^{(1)} + 1]$. Hence $l_{22} = 1$, $T_{23} = [o_1^{(1)} + 1]$. We infer $o_1^{(2)} = 1$ and $o_1^{(1)} = 2$. Put $k = f_2 - 2$. We have $k \geq 1$, $B_1^{(1)} = [k + 2]$, $A_2^{(2)} = [k + 2]$, $B_2^{(2)} = [2]$ and $A_1^{(2)} = [3]$. By Proposition 9, we get $A_1^{(1)} = [2, 3, t_k]$ and $B_2^{(2)} = t_k$. The numerical data of C is equal to $\{(2(k+1), (k+1)_2), ((k+2)_2, k+1)\}$, which coincides with the data 2 with $a = 1$, $b = k + 1$.

5.2 $B_{g_2}^{(2)} = [F'_1, T_{13}]$

If $T_{13} = \emptyset$, then this case is contained in the case 5.1. Thus we may assume $T_{13} \neq \emptyset$. We have $T_{12} = \emptyset$.

Lemma 21. *We have $g_2 = 1$, $T_{22} = \emptyset$ and $A_1^{(2)} = {}^t[F'_2, T_{23}]$. Moreover, $[T_{24}, S_3]^* = [t_{k_{12}+k_{34}-2}, k_{34} + 1, {}^tT_{14}] * t_{o_1^{(2)}}$, $B_{g_1}^{(1)} = t_{k_{12}}$.*

PROOF. Since $T_{12} = \emptyset$, $[S_1, T_{11}] = [l_{22} + 2, t_{k_{12}-1}]$ by Lemma 19. This means that $s_1 = l_{22} + 2$, $T_{11} = t_{k_{12}-1}$. By Proposition 9 and Lemma 19, $A_{g_2}^{(2)} = t_{o_{g_2}^{(2)}} * [T_{14}, k_{34} + 1, t_{k_{12}-1}]$. Suppose $g_2 = 2$. Since $A_2^{(2)} = F'_2$, we get $o_2^{(2)} = 1$, $T_{14} = \emptyset$, $k_{12} = 1$ and $f_2 = k_{34} + 2$. We have $g_1 = 1$ and $A_1^{(1)} = [T_{24}, S_3]$. By Proposition 9, $[T_{24}, S_3]^* = [l_{22} + 2, o_1^{(1)} + 1]$. By Lemma 19, T_{22} consists of (-2) -curves. By Proposition 9, we have $A_1^{(2)} = {}^tT_{23}$, $B_1^{(2)} = {}^tT_{22}$ and $T_{23} = [l_{22} + 2, t_{o_1^{(2)}-1}]$. By Lemma 19, $[T_{24}, S_3]^* = [t_{k_{34}-1}, k_{34} + 2, l_{22} + 2, t_{o_1^{(2)}-1}]$. Thus $[l_{22} + 2, o_1^{(1)} + 1] = [t_{k_{34}-1}, k_{34} + 2, l_{22} + 2, t_{o_1^{(2)}-1}]$. This shows $k_{34} = 1$. We have $[F'_1, T_{13}] = [2]$ by Lemma 19, which contradicts $T_{13} \neq \emptyset$. Hence $g_2 = 1$.

Suppose $T_{22} \neq \emptyset$. We have $T_{23} = \emptyset$ and $A_1^{(2)} = [T_{22}, F'_2]$. Since $A_1^{(2)} = t_{o_1^{(2)}} * [T_{14}, k_{34} + 1, t_{k_{12}-1}]$ and $T_{22} = t_{l_{22}}$, we infer $T_{14} = \emptyset$, $k_{12} = 1$ and $f_2 = k_{34} + 2$. We have $g_1 = 1$ and $A_1^{(1)} = [T_{24}, S_3]$. By Proposition 9,

$[T_{24}, S_3]^* = [l_{22}+2, o_1^{(1)}+1]$. By Lemma 19, $[t_{k_{34}-1}, k_{34}+2] = [l_{22}+2, o_1^{(1)}+1]$. This shows $k_{34} = 2$ and $l_{22} = 0$, which is absurd. Hence $T_{22} = \emptyset$. We get $A_1^{(2)} = {}^t[F_2', T_{23}]$. By Lemma 19, we have $[T_{24}, S_3]^* = [t_{k_{12}+k_{34}-2}, k_{34}+1, {}^tT_{14}] * t_{o_1^{(2)}}^{(1)}$ and $B_{g_1}^{(1)} = t_{k_{12}}$. \square

Case (1): $k_{34} = 1$. Suppose $T_{14} = \emptyset$. We have $g_1 = 1$ and $A_1^{(1)} = [T_{24}, S_3]$. By Lemma 19, $[T_{24}, S_3] = [t_{o_1^{(1)}-1}, k_{12}+2]$. On the other hand, $[T_{24}, S_3] = [o_1^{(2)}+1, k_{12}+1]$ by Lemma 21, which is impossible. Thus $T_{14} \neq \emptyset$. By Lemma 21, $[T_{24}, S_3] = [o_1^{(2)}+1, {}^tT_{14}^* * t_1^{*k_{12}}]$. Thus $T_{24} \neq \emptyset$. We have $g_1 = 2$ and $A_2^{(1)} = S_3$. By Lemma 19, $o_2^{(1)} = 1$ and $s_3 = k_{12}+2$. Either $A_1^{(1)} = T_{14}$ or $A_1^{(1)} = T_{24}$. If $A_1^{(1)} = T_{14}$, then $T_{14}^* = [{}^tT_{24}, o_1^{(1)}+1]$ by Proposition 9. We get $[T_{24}, S_3] = [o_1^{(2)}+1, o_1^{(1)}+1, T_{24} * t_1^{*k_{12}}]$, which is a contradiction. Hence $A_1^{(1)} = T_{24}$ and $B_1^{(1)} = {}^tT_{14}$. By Proposition 9, $T_{24} = t_{o_1^{(1)}} * {}^tT_{14}^*$. Thus $[t_{o_1^{(1)}} * {}^tT_{14}^*, S_3] = [o_1^{(2)}+1, {}^tT_{14}^* * t_1^{*k_{12}}]$. Hence $o_1^{(1)} = 1$. We have $t_{s_3-1} * [{}^tT_{14}, 2] = [t_{k_{12}}, {}^tT_{14}] * t_{o_1^{(2)}}^{(1)}$. This shows $s_3 = k_{12} + o_1^{(2)}$ and $o_1^{(2)} = 2$. Thus $[{}^tT_{14}^*, 2] = [2, {}^tT_{14}^*]$. There exists a positive integer l such that $T_{14}^* = t_l$ by Lemma 5 (iii). Write $k = k_{12}$. We have $B_2^{(1)} = t_k$, $A_2^{(1)} = [k+2]$ and $B_1^{(1)} = [l+1]$. By Lemma 19, $B_1^{(2)} = [k+2, t_{l-1}]$. By Proposition 9, we have $A_1^{(1)} = [3, t_{l-1}]$ and $A_1^{(2)} = [2, l+2, t_k]$. The numerical data of C is equal to $\{((l+1)(k+1), l(k+1), (k+1)_l), ((l(k+1)+1)_2, (k+1)_l)\}$, which coincides with the data 2 with $a = l$, $b = k+1$.

Case (2): $k_{34} > 1$, $T_{14} = \emptyset$. We have $g_1 = 1$ and $A_1^{(1)} = [T_{24}, S_3]$. By Proposition 9 and Lemma 21, $[T_{24}, S_3]^* = [B_1^{(1)}, o_1^{(1)}+1] = [t_{k_{12}}, o_1^{(1)}+1]$. On the other hand, $[T_{24}, S_3]^* = [t_{k_{12}+k_{34}-2}, k_{34}+2, t_{o_1^{(2)}-1}]$ by Lemma 21. Hence $k_{34} = 2$, $o_1^{(2)} = 1$ and $o_1^{(1)} = 3$. Write $k = k_{12}$. We have $B_1^{(1)} = t_k$. By Lemma 19, $B_1^{(2)} = [k+1, 2]$. By Proposition 9, we have $A_1^{(1)} = [2, 2, k+2]$ and $A_1^{(2)} = [4, t_{k-1}]$. The numerical data of C is equal to $\{((k+1)_3), (2k+1, k_2)\}$, which coincides with the data 1 with $a = 1$, $b = k+1$.

Case (3): $k_{34} > 1$, $T_{14} \neq \emptyset$. By Lemma 21, $[T_{24}, S_3] = [o_1^{(2)}+1, {}^tT_{14}^* * t_{k_{34}-1}, k_{12}+k_{34}]$. We have $T_{24} = [o_1^{(2)}+1, {}^tT_{14}^* * t_{k_{34}-1}]$ and $s_3 = k_{12} + k_{34}$. We infer $T_{24} \neq \emptyset$ and $g_1 = 2$. Since $A_2^{(1)} = S_3$, we get $o_2^{(1)} = 1$ and $k_{34} = 2$ by Lemma 19. Either $B_1^{(1)} = {}^tT_{14}$ or $B_1^{(1)} = {}^tT_{24}$. If $B_1^{(1)} = {}^tT_{24}$, then $T_{14}^* = [o_1^{(1)}+1, T_{24}]$ by Proposition 9. Thus $T_{24} = [o_1^{(2)}+1, o_1^{(1)}+1, T_{24} * t_1]$, which is absurd. Hence $B_1^{(1)} = {}^tT_{14}$. By Proposition 9, $T_{24} = A_1^{(1)} = t_{o_1^{(1)}} * {}^tT_{14}^*$. This means that $o_1^{(1)} = 2$, $o_1^{(2)} = 1$ and $t_1 * {}^tT_{14}^* = {}^tT_{14}^* * t_1$. Thus $[T_{14}, 2] = [2, T_{14}]$. There exists a positive integer l such that $T_{14} = t_l$. Write $k = k_{12}$. We have $B_2^{(1)} = t_k$, $A_2^{(1)} = [k+2]$, $B_1^{(1)} = t_l$. By Lemma 19, $B_1^{(2)} = [k+1, l+2]$.

By Proposition 9, we have $A_1^{(1)} = [2, l+2]$ and $A_1^{(2)} = [3, t_{l-1}, 3, t_{k-1}]$. The numerical data of C is equal to $\{(((l+1)(k+1))_2, (k+1)_{l+1}), ((l+1)(k+1)+k, l(k+1)+k, (k+1)_l, k)\}$, which coincides with the data 1 with $a = l+1$, $b = k+1$.

5.3 $B_{g_2}^{(2)} = {}^t[T_{22}, F'_2]$

We have $T_{23} = \emptyset$. We may assume $T_{11} \neq \emptyset$ because this case is contained in the case 5.1 if $T_{11} = \emptyset$.

Lemma 22. *We have $g_1 = 1$, $T_{24} = \emptyset$, $A_1^{(1)} = [T_{14}, S_3]$, $f_2 = 2$, $s_3 = k_{34}+1$, $f_1 = l_{22}+3$, and $[F'_1, T_{13}]^* = t_{o_1^{(1)}} * [T_{12}, k_{12}+1, t_{l_{22}+k_{12}-1}]$.*

PROOF. By Lemma 19, we have $[T_{24}, S_3] = [t_{f_2-2}, k_{34}+1]$. This shows that $T_{24} = t_{f_2-2}$ and $s_3 = k_{34}+1$. Suppose $g_1 = 2$. Since $A_2^{(1)} = S_3$, we have $[k_{34}+1] = t_{o_2^{(1)}} * [T_{12}, k_{12}+1, t_{l_{22}}]$ by Lemma 19. This means that $o_2^{(1)} = 1$, $T_{12} = \emptyset$, $l_{22} = 0$ and $k_{34} = k_{12}+1$. We infer $g_2 = 1$ and $A_1^{(2)} = {}^t[F'_1, T_{13}]$. By Proposition 9, $[F'_1, T_{13}]^* = {}^tA_1^{(2)*} = [o_1^{(2)}+1, F'_2]$. By Lemma 19, $[o_1^{(2)}+1, F'_2] = [T_{14}, k_{12}+2, t_{k_{12}-1}]$. Because $T_{14} \neq \emptyset$, we have $k_{12} = 1$. By Lemma 19, we obtain $[S_1, T_{11}] = [2]$, which contradicts $T_{11} \neq \emptyset$. Hence $g_1 = 1$.

Suppose $T_{24} \neq \emptyset$. We have $T_{14} = \emptyset$ and $A_1^{(1)} = [T_{24}, S_3] = [t_{f_2-2}, k_{34}+1]$. Thus $f_2 > 2$. By Lemma 19, $[F'_1, T_{13}] = [k_{12}+1, t_{k_{34}-1}]$. This shows $f_1 = k_{12}+1$. By Proposition 9, $A_{g_2}^{(2)} = [t_{o_{g_2}^{(2)}-1}, l_{22}+3, t_{f_2-2}]$. Since $A_{g_2}^{(2)} = {}^t[F'_1, \dots]$, we have $f_1 = 2$ and $k_{12} = 1$. By Lemma 19, $A_1^{(1)} = t_{o_1^{(1)}} * [T_{12}, t_{l_{22}+1}]$. Thus $[t_{f_2-2}, k_{34}+1] = t_{o_1^{(1)}} * [T_{12}, t_{l_{22}+1}]$. If $T_{12} \neq \emptyset$ or $l_{22} > 0$, then $k_{34} = 1$ and $t_{f_2-2} = t_{o_1^{(1)}} * [T_{12}, t_{l_{22}}]$, which is impossible. Thus $T_{12} = \emptyset$ and $l_{22} = 0$. By Lemma 19, $[S_1, T_{11}] = [2]$, which contradicts $T_{11} \neq \emptyset$. Hence $T_{24} = \emptyset$ and $A_1^{(1)} = [T_{14}, S_3]$. We have $f_2 = 2$. By Proposition 9, $A_{g_2}^{(2)} = [t_{o_{g_2}^{(2)}-1}, l_{22}+3]$. This shows $f_1 = l_{22}+3$. By Lemma 19, $[F_1, T_{13}]^* = t_{o_1^{(1)}} * [T_{12}, k_{12}+1, t_{l_{22}+k_{12}-1}]$. \square

Case (1): $g_2 = 1$. If $T_{13} = \emptyset$, then $t_{f_1-1} = t_{o_1^{(1)}} * [T_{12}, k_{12}+1, t_{l_{22}+k_{12}-1}]$ by Lemma 22, which is absurd. Thus $T_{13} \neq \emptyset$. We have $T_{12} = \emptyset$ and $A_1^{(2)} = {}^t[F'_1, T_{13}]$. By Lemma 22, ${}^tA_1^{(2)*} = t_{o_1^{(1)}} * [k_{12}+1, t_{l_{22}+k_{12}-1}]$. By Proposition 9, ${}^tA_1^{(2)*} = [o_1^{(2)}+1, t_{l_{22}+1}]$. It follows that $o_1^{(1)}+k_{12} = 3$. By Lemma 19, $[S_1, T_{11}] = [l_{22}+2, t_{k_{12}-1}]$. Since $T_{11} \neq \emptyset$, we see $k_{12} = 2$, $o_1^{(1)} = 1$ and $o_1^{(2)} = 3$. Put $k = l_{22}+1$. We have $k \geq 1$, $B_1^{(1)} = [k+1, 2]$ and $B_1^{(2)} = t_k$. By Proposition 9, we get $A_1^{(1)} = [4, t_{k-1}]$ and $A_1^{(2)} = [2, 2, k+2]$. The numerical data of C is equal to $\{(2k+1, k_2), ((k+1)_3)\}$, which coincides with the data 1 with $a = 1$, $b = k+1$.

Case (2): $g_2 = 2$. We have $A_2^{(2)} = F'_1$, $T_{12} \neq \emptyset$ and $T_{13} \neq \emptyset$.

Lemma 23. *We have $B_1^{(2)} = {}^tT_{12}$, $A_1^{(2)} = {}^tT_{13}$, $[o_1^{(2)} + 1, T_{12} * t_{l_{22}+2}] = [t_{o_1^{(1)}} * T_{12}, k_{12} + 1, t_{l_{22}+k_{12}-1}]$ and $3 = o_1^{(1)} + k_{12}$.*

PROOF. Either $B_1^{(2)} = {}^tT_{12}$ or $B_1^{(2)} = T_{13}$. Suppose $B_1^{(2)} = T_{13}$. By Proposition 9, $T_{12} = A_1^{(2)} = t_{o_1^{(2)}} * T_{13}^*$. By Lemma 22, $[l_{22} + 3, T_{13}] = [t_{o_1^{(1)}} * t_{o_1^{(2)}} * T_{13}^*, k_{12} + 1, t_{l_{22}+k_{12}-1}]^* = t_1^{*l_{22}+k_{12}-1} * t_{k_{12}} * [T_{13}, o_1^{(2)} + 1, o_1^{(1)} + 1]$, which is impossible. Thus $B_1^{(2)} = {}^tT_{12}$ and $A_1^{(2)} = {}^tT_{13}$. By Proposition 9, $T_{13}^* = {}^tA_1^{(2)*} = [o_1^{(2)} + 1, T_{12}]$. By Lemma 22, $[o_1^{(2)} + 1, T_{12} * t_{l_{22}+2}] = [t_{o_1^{(1)}} * T_{12}, k_{12} + 1, t_{l_{22}+k_{12}-1}]$. Hence $3 = o_1^{(1)} + k_{12}$. \square

Case (2-1): $k_{12} = 1$. By Lemma 23, we have $o_1^{(1)} = 2$, $o_1^{(2)} = 1$ and $T_{12} * t_1 = t_1 * T_{12}$. Thus $[2, T_{12}^*] = [T_{12}^*, 2]$. There exists a positive integer l' such that $T_{12}^* = t_{l'}$. Hence $T_{12} = [l' + 1]$. Put $l = l' - 1$ and $k = l_{22} + 1$. By Lemma 19, $B_1^{(1)} = [k + 2, t_l]$. Since $T_{11} \neq \emptyset$, we see $l \geq 1$. We have $B_2^{(2)} = t_k$, $A_2^{(2)} = [k + 2]$ and $B_1^{(2)} = [l + 2]$. By Proposition 9, we see $A_1^{(1)} = [2, l + 3, t_k]$ and $A_1^{(2)} = [3, t_l]$. The numerical data of C is equal to $\{((l+1)(k+1)+1)_2, (k+1)_{l+1}, ((l+2)(k+1), (l+1)(k+1), (k+1)_{l+1})\}$, which coincides with the data 2 with $a = l + 1$, $b = k + 1$.

Case (2-2): $k_{12} = 2$. By Lemma 23, $o_1^{(1)} = 1$ and $[o_1^{(2)}, T_{12}] = [T_{12}, 2]$. We have $o_1^{(2)} \geq 2$. Since $T_{12}^* * t_{o_1^{(2)}-1} = t_1 * T_{12}^*$, we see $o_1^{(2)} = 2$. There exists a positive integer l such that $T_{12} = t_l$. Put $k = l_{22} + 1$. We have $B_2^{(2)} = t_k$, $A_2^{(2)} = [k + 2]$ and $B_1^{(2)} = t_l$. By Lemma 19, $B_1^{(1)} = [k + 1, l + 2]$. By Proposition 9, we have $A_1^{(1)} = [3, t_{l-1}, 3, t_{k-1}]$ and $A_1^{(2)} = [2, l + 2]$. The numerical data of C is equal to $\{((l+1)(k+1) + k, l(k+1) + k, (k+1)_l, k), (((l+1)(k+1))_2, (k+1)_{l+1})\}$, which coincides with the data 1 with $a = l + 1$, $b = k + 1$.

5.4 $B_{g_2}^{(2)} = [F'_2, T_{23}]$

We have $T_{22} = \emptyset$. We may assume $T_{11} \neq \emptyset \neq T_{23}$; otherwise this case is contained in another case.

Case (1): $g_2 = 1$. We show the following lemma.

Lemma 24. *We have $T_{12} = \emptyset$, $A_1^{(2)} = {}^t[F'_1, T_{13}]$, $f_2 = 2$, $B_{g_1}^{(1)} = t_{k_{12}}$, $k_{12} \geq 2$ and $[T_{14}, k_{34} + 1, t_{k_{12}-2}] = [o_1^{(2)} + 1, {}^tT_{23}]$.*

PROOF. Suppose $T_{12} \neq \emptyset$. We have $A_1^{(2)} = [T_{12}, F'_1]$ and $T_{13} = \emptyset$. By Lemma 19, $[T_{14}, k_{34} + 1, t_{k_{12}-1}] = t_{f_1-1}$. This shows that $k_{34} = 1$ and $T_{14} = t_{f_1-k_{12}-1}$. By Lemma 19, $A_{g_1}^{(1)} = t_{o_{g_1}^{(1)}} * [T_{12}, k_{12} + 1]$. Thus $A_{g_1}^{(1)}$

contains at least two irreducible components. It follows that $g_1 = 1$. Either $A_1^{(1)} = [T_{14}, S_3]$ or $A_1^{(1)} = [T_{24}, S_3]$. Because T_{14} consists of (-2) -curves, the latter case must occur. We infer $T_{24} = t_{o_1^{(1)}} * T_{12}$. By Proposition 9 and Lemma 19, $[T_{12}, F'_1] = A_1^{(2)} = t_{o_1^{(2)}} * [F'_2, T_{23}]^* = t_{o_1^{(2)}} * [T_{24}, S_3]$. We have $T_{12} = t_{o_1^{(2)}} * T_{24}$. Thus $T_{24} = t_{o_1^{(1)}} * t_{o_1^{(2)}} * T_{24}$, which is impossible. Hence $T_{12} = \emptyset$.

We have $A_1^{(2)} = {}^t[F'_1, T_{13}]$. By Lemma 19, $[T_{14}, k_{34} + 1, t_{k_{12}-1}] = {}^tA_1^{(2)*}$ and $B_{g_1}^{(1)} = [S_1, T_{11}] = t_{k_{12}}$. We infer $k_{12} \geq 2$. By Proposition 9, $[T_{14}, k_{34} + 1, t_{k_{12}-1}] = [o_1^{(2)} + 1, {}^tT_{23}, F'_2]$. This shows that $f_2 = 2$ and $[T_{14}, k_{34} + 1, t_{k_{12}-2}] = [o_1^{(2)} + 1, {}^tT_{23}]$. \square

Case (1-1): $g_1 = 1$. Suppose $T_{24} \neq \emptyset$. We have $T_{14} = \emptyset$ and $A_1^{(1)} = [T_{24}, S_3]$. By Lemma 19 and Lemma 24, $[t_{k_{34}}, T_{23}] = A_1^{(1)*}$. By Proposition 9, we have $[t_{k_{34}}, T_{23}] = [t_{k_{12}}, o_1^{(1)} + 1]$. By Lemma 24, $[T_{23}, o_1^{(2)} + 1] = [t_{k_{12}-2}, k_{34} + 1]$. We infer $[t_{k_{12}}, o_1^{(1)} + 1, o_1^{(2)} + 1] = [t_{k_{12}+k_{34}-2}, k_{34} + 1]$. This means that $k_{34} = 3$ and $o_1^{(1)} = 1$. By Proposition 9, $[T_{24}, S_3] = [k_{12} + 2]$, which is absurd. Thus $T_{24} = \emptyset$.

We have $A_1^{(1)} = [T_{14}, S_3]$. By Lemma 19, we get $[T_{14}, S_3] = [t_{o_1^{(1)}-1}, k_{12} + 2]$. Hence $s_3 = k_{12} + 2$ and $T_{14} = t_{o_1^{(1)}-1}$. By Proposition 9 and Lemma 24, $[F'_2, T_{23}, o_1^{(2)} + 1] = A_1^{(2)*} = {}^t[F'_1, T_{13}]^*$. By Lemma 19, $[F'_2, T_{23}, o_1^{(2)} + 1] = [t_{k_{12}-1}, k_{34} + 1, t_{o_1^{(1)}-1}]$. Since $k_{12} \geq 2$, we see $[T_{23}, o_1^{(2)} + 1] = [t_{k_{12}-2}, k_{34} + 1, t_{o_1^{(1)}-1}]$. By Lemma 19, $[t_{k_{34}-1}, F'_2, T_{23}] = t_{s_3-1} = t_{k_{12}+1}$. Thus $T_{23} = t_{k_{12}-k_{34}+1}$. Hence $[t_{k_{12}-k_{34}+1}, o_1^{(2)} + 1] = [t_{k_{12}-2}, k_{34} + 1, t_{o_1^{(1)}-1}]$. We infer $4 = k_{34} + o_1^{(1)}$.

Case (1-1_a): $k_{34} = 1$. We have $o_1^{(1)} = 3$, $o_1^{(2)} = 1$ and $[F'_2, T_{23}] = t_{k_{12}+1}$. Put $k = k_{12} - 1$. By Lemma 24, $k \geq 1$ and $B_1^{(1)} = t_{k+1}$. We have $B_1^{(2)} = t_{k+2}$. By Proposition 9, $A_1^{(1)} = [2, 2, k + 3]$ and $A_1^{(2)} = [k + 4]$. The numerical data of C is equal to $\{(k + 3), ((k + 2)_3)\}$, which coincides with the data 3 with $a = 1$, $b = k + 2$.

Case (1-1_b): $k_{34} > 1$. If $o_1^{(1)} = 2$, then $[t_{k_{12}-k_{34}+1}, o_1^{(2)} + 1] = [t_{k_{12}-2}, 3, 2]$, which is impossible. We have $o_1^{(1)} = 1$, $k_{34} = 3$ and $o_1^{(2)} = 3$. Put $k = k_{12} - 2$. Since $[F'_2, T_{23}] = t_{k+1}$, we see $k \geq 1$. We have $B_1^{(1)} = t_{k+2}$ and $B_1^{(2)} = t_{k+1}$. By Proposition 9, we obtain $A_1^{(1)} = [k + 4]$ and $A_1^{(2)} = [2, 2, k + 3]$. The numerical data of C is equal to $\{((k + 2)_3), (k + 3)\}$, which coincides with the data 3 with $a = 1$, $b = k + 2$.

Case (1-2): $g_1 = 2$. We have $A_2^{(1)} = S_3$, $T_{14} \neq \emptyset$ and $T_{24} \neq \emptyset$.

Lemma 25. *We have $o_2^{(1)} = 1$, $s_3 = k_{12} + 2$, $A_1^{(1)} = T_{14}$, $B_1^{(1)} = {}^tT_{24}$,*

$$[t_{k_{12}+k_{34}-2}, k_{34} + 1, T_{24}^* * t_{o_1^{(1)}}] = [t_{k_{12}+1} * T_{24}^*, o_1^{(2)} + 1], k_{34} + o_1^{(1)} = 3.$$

PROOF. By Lemma 19, we get $o_2^{(1)} = 1$, $s_3 = k_{12} + 2$ and $[t_{k_{34}}, T_{23}] = t_{k_{12}+1} * T_{24}^*$. Either $B_1^{(1)} = {}^tT_{24}$ or $B_1^{(1)} = {}^tT_{14}$. Suppose $B_1^{(1)} = {}^tT_{14}$. We have $A_1^{(1)} = T_{24}$. By Proposition 9, $[t_{k_{34}}, T_{23}] = [t_{k_{12}+1} * {}^tT_{14}, o_1^{(1)} + 1]$. By Lemma 24, we have $[T_{14}, k_{34} + 1, t_{k_{12}+k_{34}-2}] = [o_1^{(2)} + 1, {}^tT_{23}, t_{k_{34}}] = [o_1^{(2)} + 1, o_1^{(1)} + 1, T_{14} * t_{k_{12}+1}]$. Thus $k_{34} = 3$. It follows that $[T_{14}, 4, 2] = [o_1^{(2)} + 1, o_1^{(1)} + 1, T_{14} * t_1]$, which is impossible. Hence $A_1^{(1)} = T_{14}$, $B_1^{(1)} = {}^tT_{24}$. By Proposition 9, $T_{14} = t_{o_1^{(1)}} * {}^tT_{24}^*$. By Lemma 24, $[T_{23}, o_1^{(2)} + 1] = [t_{k_{12}-2}, k_{34} + 1, T_{24}^* * t_{o_1^{(1)}}]$. Thus $[t_{k_{12}+1} * T_{24}^*, o_1^{(2)} + 1] = [t_{k_{12}+k_{34}-2}, k_{34} + 1, T_{24}^* * t_{o_1^{(1)}}]$. Hence $3 = k_{34} + o_1^{(1)}$. \square

Case (1-2_a): $k_{34} = 1$. By Lemma 25, we have $o_1^{(1)} = 2$, $o_1^{(2)} = 1$ and $[t_{k_{12}+1} * T_{24}^*, 2] = [t_{k_{12}}, T_{24}^* * t_2]$. Thus $t_1 * T_{24}^* = T_{24}^* * t_1$. Hence $[T_{24}, 2] = [2, T_{24}]$. There exists a positive integer l such that $T_{24} = t_l$. Put $k = k_{12} - 1$. By Lemma 24, we have $k \geq 1$, $[S_1, T_{11}] = B_2^{(1)} = t_{k+1}$. By Lemma 25, $A_2^{(1)} = [k + 3]$. Since $B_1^{(1)} = t_l$, we get $A_1^{(1)} = [2, l + 2]$ by Proposition 9. By Lemma 19, we infer $B_1^{(2)} = [t_{k+1}, l + 2]$. By Proposition 9, $A_1^{(2)} = [3, t_{l-1}, k + 3]$. It follows that the numerical data of C is equal to $\{((l + 1)(k + 2))_2, (k + 2)_{l+1}, ((l + 1)(k + 2) + 1, l(k + 2) + 1, (k + 2)_l)\}$, which coincides with the data 3 with $a = l + 1$, $b = k + 2$.

Case (1-2_b): $k_{34} = 2$. By Lemma 25, we have $o_1^{(1)} = 1$ and $[t_{k_{12}}, 3, T_{24}^* * t_1] = [t_{k_{12}+1} * T_{24}^*, o_1^{(2)} + 1]$. Thus $[2, T_{24}^*] = [T_{24}^*, o_1^{(2)}]$. Hence $T_{24} * t_1 = t_{o_1^{(2)}-1} * T_{24}$. This shows that $o_1^{(2)} = 2$ and $[2, T_{24}^*] = [T_{24}^*, 2]$. There exists a positive integer l such that $T_{24}^* = t_l$. We have $T_{24} = [l + 1]$. Put $k = k_{12} - 1$. By Lemma 24, we see $k \geq 1$, $[S_1, T_{11}] = B_2^{(1)} = t_{k+1}$. We have $A_2^{(1)} = [k + 3]$ by Lemma 25. Since $B_1^{(1)} = [l + 1]$, we get $A_1^{(1)} = [3, t_{l-1}]$ by Proposition 9. By Lemma 19, we infer $B_1^{(2)} = [t_k, 3, t_{l-1}]$. By Proposition 9, $A_1^{(2)} = [2, l + 2, k + 2]$. It follows that the numerical data of C is equal to $\{((l + 1)(k + 2), l(k + 2), (k + 2)_l), ((l(k + 2) + k + 1)_2, (k + 2)_l, k + 1)\}$, which coincides with the data 4 with $a = l$, $b = k + 2$.

Case (2): $g_2 = 2$. We have $T_{12} \neq \emptyset \neq T_{13}$ and $A_2^{(2)} = F'_1$.

Lemma 26. *The following assertions hold.*

- (i) $g_1 = 1$, $T_{24} = \emptyset$, $A_1^{(1)} = [T_{14}, S_3]$, $B_1^{(2)} = {}^tT_{12}$, $A_1^{(2)} = {}^tT_{13}$.
- (ii) $k_{12} \geq 2$, $f_1 \geq 4$, $f_2 = 2$, $s_3 = k_{12} + 1 = k_{34} + f_1 - 2$, $k_{34} + o_1^{(1)} = 3$, $o_1^{(2)} = k_{34}$, $o_2^{(2)} = 1$.

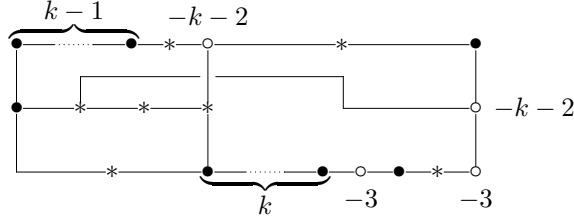
$$(iii) \quad T_{13} = T_{12}^* * t_{o_1^{(2)}}, \quad T_{14} = t_{o_1^{(1)}} * T_{12}, \quad T_{23} = t_{f_1-3}, \quad [F'_1, T_{12}^* * t_{o_1^{(2)}}] = [t_1^{*k_{12}-1} * t_{k_{34}} * T_{12}^*, o_1^{(1)} + 1].$$

PROOF. By Proposition 9, $[F'_2, T_{23}, o_2^{(2)} + 1] = t_{f_1-1}$. This shows that $f_1 \geq 4$, $f_2 = 2$, $o_2^{(2)} = 1$ and $T_{23} = t_{f_1-3}$. By Lemma 19, $[T_{24}, S_3] = [k_{34} + f_1 - 2]$. We have $T_{24} = \emptyset$ and $s_3 = k_{34} + f_1 - 2$. It follows that $g_1 = 1$ and $A_1^{(1)} = [T_{14}, S_3]$. By Lemma 19, $[T_{14}, S_3] = t_{o_1^{(1)}} * [T_{12}, k_{12} + 1]$. Since $T_{12} \neq \emptyset$, we obtain $\emptyset \neq t_{o_1^{(1)}} * T_{12} = T_{14}$ and $k_{12} + 1 = s_3 = k_{34} + f_1 - 2$. We have $k_{12} \geq 2$ and $T_{14}^* = [T_{12}^*, o_1^{(1)} + 1]$. By Lemma 19, $[F'_1, T_{13}] = [t_1^{*k_{12}-1} * t_{k_{34}} * T_{12}^*, o_1^{(1)} + 1]$. Either $B_1^{(2)} = T_{13}$ or $B_1^{(2)} = {}^tT_{12}$. If $B_1^{(2)} = T_{13}$, then $A_1^{(2)} = T_{12}$. By Proposition 9, $[T_{13}, o_1^{(2)} + 1] = T_{12}^*$. Hence $[F'_1, T_{13}] = [t_1^{*k_{12}-1} * t_{k_{34}} * T_{13}, o_1^{(2)} + 1, o_1^{(1)} + 1]$, which is impossible. Thus $B_1^{(2)} = {}^tT_{12}$, $A_1^{(2)} = {}^tT_{13}$. By Proposition 9, $T_{13} = T_{12}^* * t_{o_1^{(2)}}$. We have $[F'_1, T_{12}^* * t_{o_1^{(2)}}] = [t_1^{*k_{12}-1} * t_{k_{34}} * T_{12}^*, o_1^{(1)} + 1]$. This means that $o_1^{(2)} = k_{34}$ and $[o_1^{(2)} + 1, T_{12} * t_{f_1-1}] = [t_{o_1^{(1)}} * T_{12}, k_{34} + 1, t_{k_{12}-1}]$. We have $f_1 = o_1^{(1)} + k_{12}$. Hence $k_{34} + o_1^{(1)} = 3$. \square

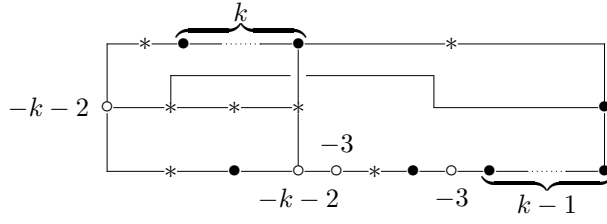
Case (2-1): $k_{34} = 1$. By Lemma 26, $k_{12} = f_1 - 2$, $o_1^{(1)} = 2$, $o_1^{(2)} = 1$, $[F'_1, T_{12}^* * t_1] = [t_1^{*k_{12}} * T_{12}^*, 3]$. We have $[2, T_{12} * t_{f_1-1}] = [t_2 * T_{12}, t_{f_1-2}]$. This shows $T_{12} * t_1 = t_1 * T_{12}$. There exists a positive integer l such that $T_{12} = [l + 1]$. Put $k = k_{12} - 1$. By Lemma 26, we have $k \geq 1$, $B_1^{(2)} = {}^tT_{12} = [l + 1]$. Furthermore, we see $A_1^{(2)} = {}^tT_{13} = [3, t_{l-1}]$, $A_2^{(2)} = [k + 3]$, $A_1^{(1)} = [T_{14}, S_3] = [2, l + 2, k + 2]$, $B_2^{(2)} = [F'_2, T_{23}] = t_{k+1}$. By Proposition 9, we obtain $[B_1^{(1)}, 3] = A_1^{(1)*} = t_{k+1} * t_{l+1} * t_1$. Hence $B_1^{(1)} = [t_k, 3, t_{l-1}]$. The numerical data of C is equal to $\{((l(k+2) + k + 1)_2, (k+2)_l, k+1), ((l+1)(k+2), l(k+2), (k+2)_l)\}$, which coincides with the data 4 with $a = l$, $b = k + 2$.

Case (2-2): $k_{34} > 1$. By Lemma 26, we get $k_{34} = 2$, $o_1^{(1)} = 1$, $o_1^{(2)} = 2$, $s_3 = k_{12} + 1 = f_1 \geq 4$, $[F'_1, T_{12}^* * t_2] = [k_{12} + 1, t_1 * T_{12}^*, 2]$. We have $T_{12}^* * t_1 = t_1 * T_{12}^*$. There exists a positive integer l such that $T_{12}^* = [l + 1]$. Put $k = k_{12} - 2$. We have $k \geq 1$, $B_1^{(2)} = {}^tT_{12} = t_l$. Moreover, we see $A_1^{(2)} = {}^tT_{13} = [2, l + 2]$, $A_2^{(2)} = [k + 3]$, $A_1^{(1)} = [T_{14}, S_3] = [3, t_{l-1}, k + 3]$, $B_2^{(2)} = [F'_2, T_{23}] = t_{k+1}$. By Proposition 9, we obtain $[B_1^{(1)}, 2] = A_1^{(1)*} = t_{k+2} * [l + 1, 2]$. Hence $B_1^{(1)} = [t_{k+1}, l + 2]$. The numerical data of C is equal to $\{((l+1)(k+2) + 1, l(k+2) + 1, (k+2)_l), (((l+1)(k+2))_2, (k+2)_{l+1})\}$, which coincides with the data 3 with $a = l + 1$, $b = k + 2$.

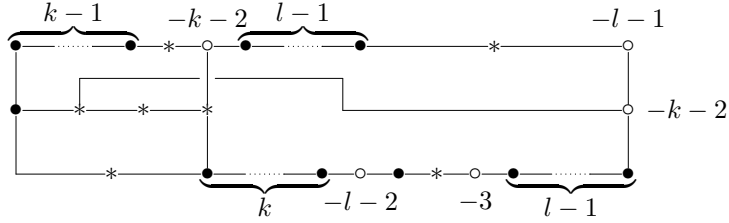
We list the dual graphs of $D + E_1 + E_2$ in Figure 4. We prove the converse assertion of Theorem 2. Let Γ be one of the weighted dual graphs in Figure 4. It follows from [Fu, Proposition 4.7] that the sub-graphs F_0 , F_1 and F_2 of Γ can be contracted to three disjoint 0-curves. After the contraction, S_1 , S_2 and S_3 become disjoint 0-curves and meet with each curve F_i transversally.



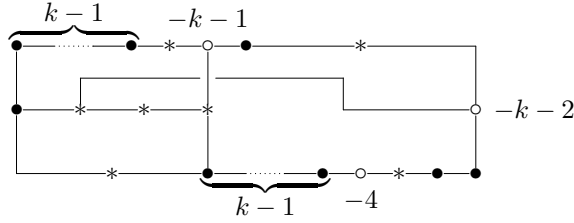
$$(1-1) B_{g_2}^{(2)} = {}^t[T_{12}, F'_1], g_2 = 1$$



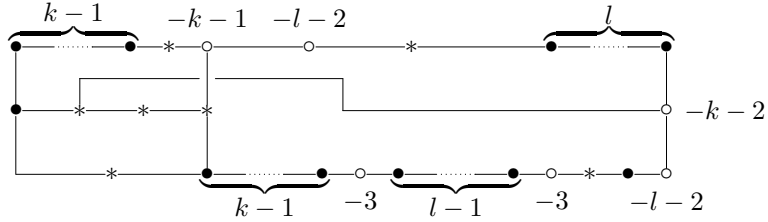
$$(1-2) B_{g_2}^{(2)} = {}^t[T_{12}, F'_1], g_2 = 2$$



$$(2-1) B_{g_2}^{(2)} = [F'_1, T_{13}], k_{34} = 1$$

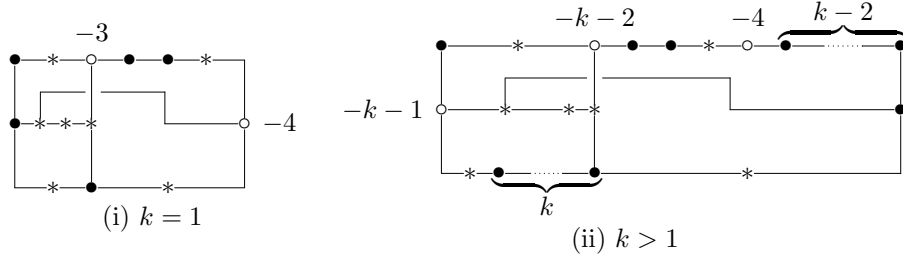


$$(2-2) B_{g_2}^{(2)} = [F'_1, T_{13}], k_{34} > 1, T_{14} = \emptyset$$



$$(2-3) B_{g_2}^{(2)} = [F'_1, T_{13}], k_{34} > 1, T_{14} \neq \emptyset$$

Figure 4: The dual graph of $D + E_1 + E_2$



$$(3-1) B_{g_2}^{(2)} = {}^t[T_{22}, F'_2], g_2 = 1$$

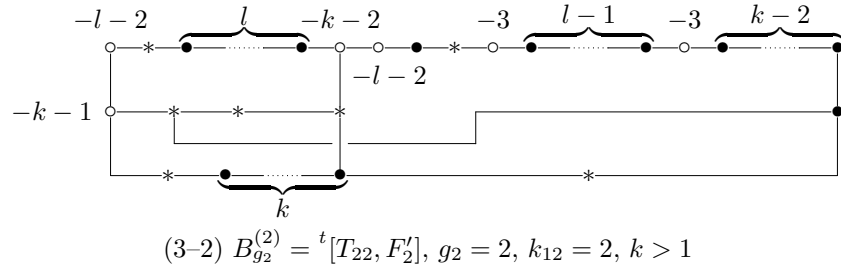
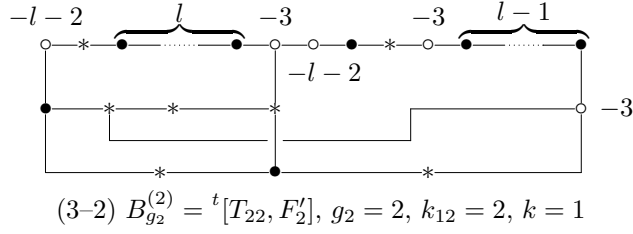
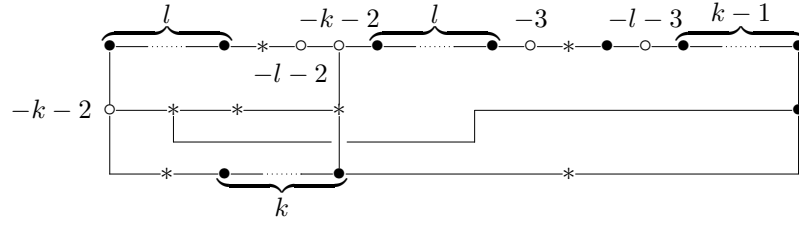
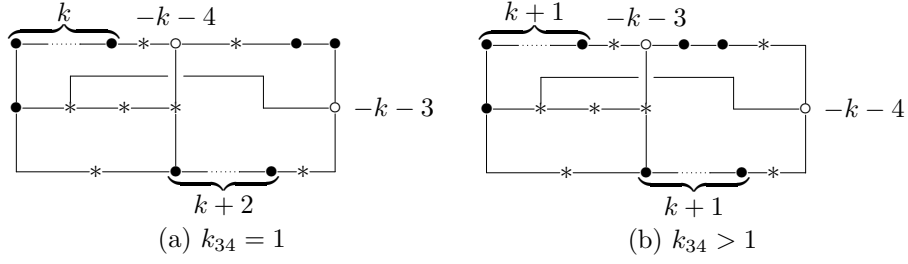
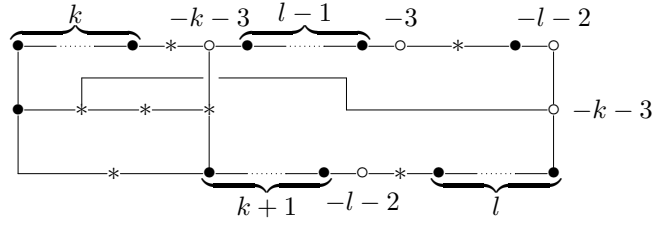


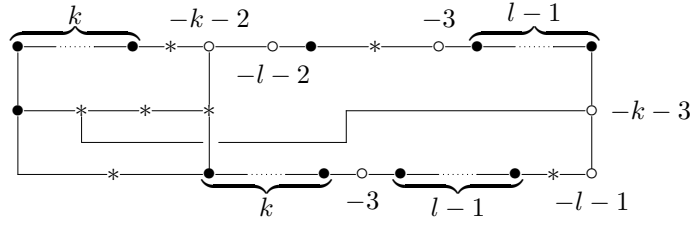
Figure 4: The dual graph of $D + E_1 + E_2$ — continued



$$(4-1-1) \ B_{g_2}^{(2)} = [F'_2, T_{23}], \ g_2 = 1, \ g_1 = 1$$

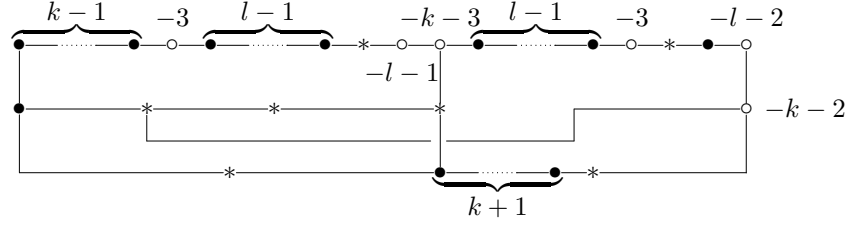


$$(4-1-2_a) \ B_{g_2}^{(2)} = [F'_2, T_{23}], \ g_2 = 1, \ g_1 = 2, \ k_{34} = 1$$

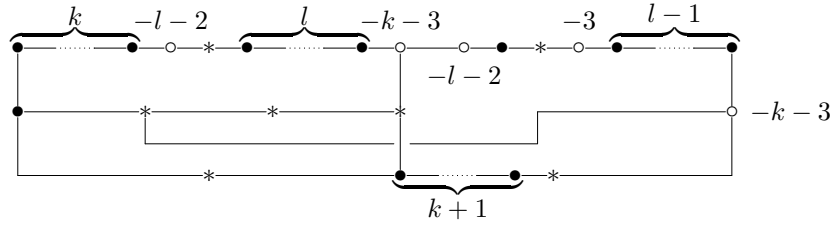


$$(4-1-2_b) \ B_{g_2}^{(2)} = [F'_2, T_{23}], \ g_2 = 1, \ g_1 = 2, \ k_{34} = 2$$

Figure 4: The dual graph of $D + E_1 + E_2$ — continued



$$(4-2-1) B_{g_2}^{(2)} = [F'_2, T_{23}], g_2 = 2, g_1 = 2, k_{34} = 1$$



$$(4-2-2) B_{g_2}^{(2)} = [F'_2, T_{23}], g_2 = 2, g_1 = 2, k_{34} > 1$$

Figure 4: The dual graph of $D + E_1 + E_2$ — continued

Thus Γ can be realized by blow-ups over three sections and fibers of Σ_0 . By Lemma 6, $\Gamma - E_1 - E_2 - C'$ can be contracted to two points of \mathbf{P}^2 . Hence all the numerical data in Theorem 2 can be realized as those of rational cuspidal plane curves.

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